NUMERICAL-ANALYTIC METHODS OF INVESTIGATING PERIODIC SOLUTIONS

A. M. SAMOILENKO
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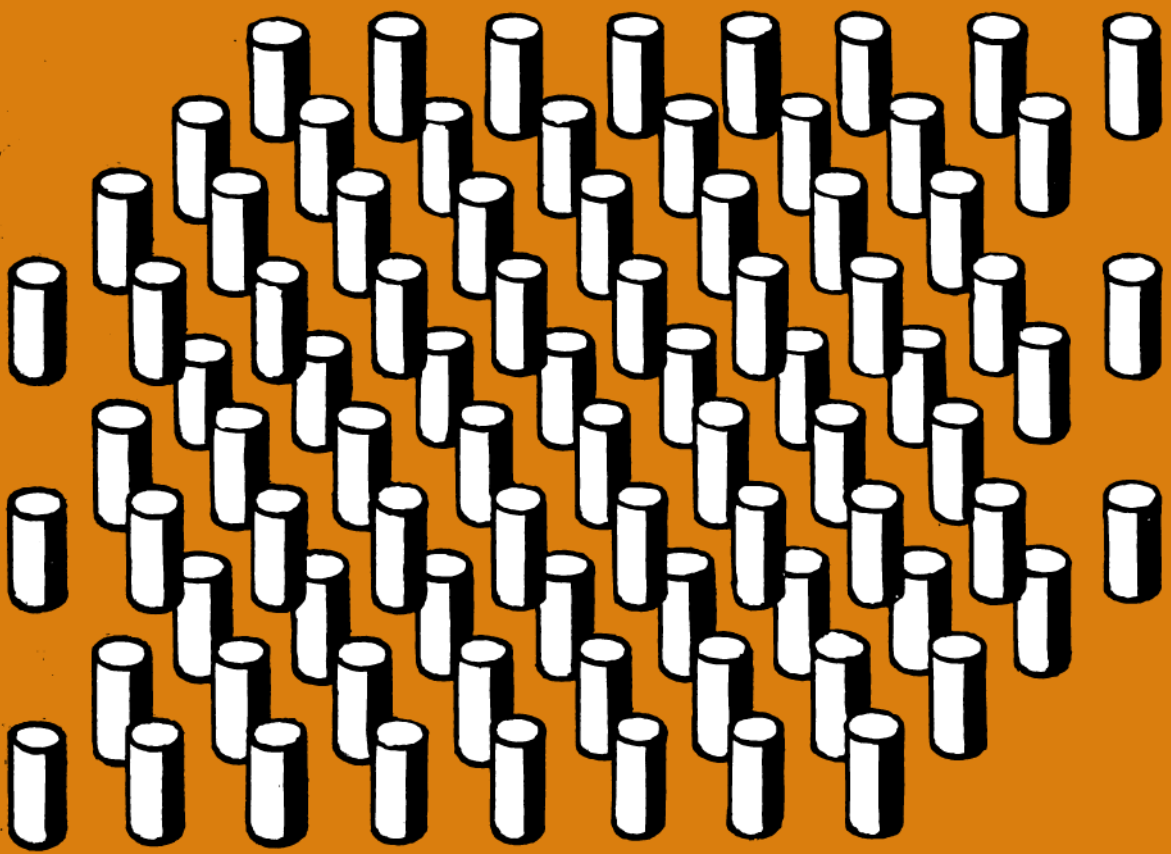

The monograph considers nonlinear periodic systems of differential equations of the general form, investigates the questions of the existence of periodic solutions and gives algorithms for finding such solutions. It substantiates the collocation method in the theory of nonlinear oscillations and gives schemes, easily realizable on a digital computer, for seeking periodic solutions.

It is intended for specialists employing methods of the theory of nonlinear oscillations and for students, post-graduates and research workers interested in the problems of this theory.

ИЗДАТЕЛЬСТВО
МИР

А. М. САМОЙЛЕНКО, Н. И. РОНТО

ЧИСЛЕННО-АНАЛИТИЧЕСКИЕ МЕТОДЫ ИССЛЕДОВАНИЯ ПЕРИОДИЧЕСКИХ РЕШЕНИЙ

ИЗДАТЕЛЬСТВО ПРИ КИЕВСКОМ ГОСУДАРСТВЕННОМ УНИВЕРСИТЕТЕ

NUMERICAL-ANALYTIC METHODS OF INVESTIGATING PERIODIC SOLUTIONS

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FOREWORD

At present there are a great number of monographs devoted to nonautonomous periodic systems of differential equations. They deal comprehensively enough with general questions of the theory of periodic solutions and present methods for their qualitative investigation and asymptotic integration as well as various algorithms for their approximate construction.

In spite of this, interest in the theory of periodic solutions of nonlinear systems of differential equations is not flagging, which may be accounted for by the importance of practical applications of this theory, on the one hand, and by difficulties in developing it for essentially nonlinear systems, on the other. The application of a variety of methods, which have already become classical, to such systems of differential equations is not fully justified.

The appearance of works establishing the limits within which these methods can be used (works on the estimation of the value of the parameter in the small-parameter method [158]) and the proof of the versatility of some of them (studies of the justification of the Galerkin method in the theory of oscillations [198, 200, 202]) are timely and important. It is, no doubt, to this kind of works that the present monograph belongs. Its first part presents a numerical-analytic method proposed by A.M. Samoilenko as early as 1965 for investigating periodic solutions of quite a definite class of systems of differential equations (T-systems), discusses the application of the method to the study of rotary and rotary-oscillatory motions and gives the results of its subsequent development as applied to similar classes of countable systems, systems of differential equations with deviating argument, integro-differential and partial differential equations. The second part is devoted to the development and foundation of the method of collocation

for constructing periodic solutions of systems of differential equations of the general form. Much attention is given to questions of the numerical realization of both methods on digital computers.

The presentation of the subject-matter is characterized by simplicity but is at the same time mathematically rigorous and accompanied by a sufficient number of examples illustrating the effectiveness of the algorithms proposed.

The book will no doubt appeal to a wide circle of specialists employing methods of the theory of nonlinear oscillations and be useful for research workers, post-graduates and students interested in the problems of this theory.

Yu.A. Mitropolsky

PREFACE

In recent years much greater interest has begun to be taken in all-round and thorough investigation of oscillatory processes arising in systems of electrical and radio engineering, vibrotechnics, celestial mechanics, instrument making, control and so on.

Quite effective as a means of investigating nonlinear oscillations are asymptotic methods of nonlinear mechanics developed in the fundamental works of N.M. Krylov, N.N. Bogolyubov and Yu.A. Mitropolsky [16, 91, 121, 122] and further elaborated by their disciples [17, 37, 39, 47, 50, 76, 105, 123, 124, 125, 127, 184, 214, 215]. These methods have been especially successful in investigating weakly nonlinear systems whose nonlinearity effect is slow to exhibit itself and can be detected by considering asymptotic expansions. New possibilities for the development and extension of asymptotic methods are offered by the appearance of digital computers performing analytic operations, such as differentiation, integration, etc. [45]. But this does not solve completely the problem of investigating even purely harmonic oscillations of systems whose nonlinearity is essential.

Many authors create and develop functional-analytic, numerical-analytic and numerical methods and schemes to investigate periodic solutions of differential equations describing these oscillations.

Functional-analytic methods [5, 25, 71, 75, 80, 82, 83, 85, 86, 87, 89, 90, 132, 144, 187, 212] make extensive use of topological concepts and are convenient means of qualitative investigation of periodic solutions, for they clear up the questions of the existence, number and stability of periodic solutions. Numerical-analytic and numerical methods [6, 18, 26, 62, 125, 128, 129, 131, 135, 150, 159, 162, 163, 171, 176, 181, 193, 195, 197, 198, 200, 202, 216, 223], owing to the great possibilities of exploiting digital computers,

are becoming a versatile means of the finding and approximate construction of periodic solutions.

The monograph being presented to the reader strikes the balance of the authors' invesigations devoted to the development and application of numerical-analytic methods for periodic systems of nonlinear differential equations. Chapter I contains the results obtained by A.M. Samoilenko and Chapter II those obtained by N.I. Ronto.

We find it a pleasant duty to express our deep gratitude to Yu.A. Mitropolsky, Academician of the Academy of Sciences of the Ukrainian SSR, for discussion of the work and constant interest he took in it.

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I

NUMERICAL-ANALYTIC METHOD FOR INVESTIGATING NONLINEAR *T*-SYSTEMS

1. AN ABSTRACT SCHEME OF THE METHOD

Consider a method that should have been more naturally called a method of successive periodic approximations but which its author happened to call a *numerical-analytic method for investigating periodic solutions* [161, 162], and the name has subsequently taken root to an extent. Its idea is simple and, as has been shown by numerous investigations [77, 102, 103, 105-109, 128, 135, 166, 173, 183-186, 190-194, 196, 209-211], convenient for finding harmonic oscillations arising in various systems described by ordinary differential equations, equations with retarded argument, with impulsive action, integro-differential and partial differential equations.

These investigations being sufficiently versatile, it seems useful to present a general scheme of the method in abstract form.

So let $C(D)$ be a space of continuous vector functions u of the variables $(x_1, x_2, \ldots, x_n) = x$ in a domain D. We denote by $\Pi(D)$ a subspace of $C(D)$ and, having in mind the final goal of applying the method, i. e. the finding of periodic solutions, call it a *space of periodic functions.*

Let A be an operator defined on $C(D)$ and having values in $AC(D) \subset C(D)$. The operator A gives the equation

$$u = Au \tag{1.1}$$

and it is required to find its periodic solution, i. e. a solution belonging to the space $\Pi(D)$. If the operator A transformed periodic functions back into periodic functions, then using the general fixed-point theorems one could solve the problem in hand by studying the given operator in $\Pi(D)$. However, for many equations of the form (1.1) the operator A tends to transform periodic functions into the functions of $C(D)$, which, generally

speaking, are not periodic. Therefore, it is impossible to find periodic solutions of equation (1.1) by considering it in $\Pi(D)$. As a consequence, besides A, one has to introduce in some way or other the operator $\tilde{A}$ whose range and domain is $\Pi(D)$.

It is natural that the operator A should be such that

(1) any periodic solution of equation (1.1) is a periodic solution of the equation

$$v = \tilde{A}v \tag{1.2}$$

(2) each periodic solution of equation (1.2), obeying some additional condition

$$Bv = 0 \tag{1.3}$$

which is given on the solutions of equation (1.2) and satisfied by all periodic solutions of the original equation, is a periodic solution of equation (1.1).

Since the operator $\tilde{A}$ acts in the space $\Pi(D)$, periodic solutions of equation (1.2) can be found by various iterative methods. The numerical-analytic scheme developed uses as such a method of successive approximations in which, from a suitably chosen zero approximation u_0 possessing certain arbitrariness (dependence on a parameter or arbitrary function), subsequent approximations are defined by the formula

$$u_{i+1} = \tilde{A}u_i, \qquad i = 0, 1, 2, \ldots \tag{1.4}$$

while an arbitrary choice of u_0 serves to satisfy condition (1.3). Here the additional condition (1.3) can be satisfied by the limiting function $u^0 = \lim_{i \to \infty} u_i$ or any iteration starting from some number.

The convergence of the successive approximations (1.4) generates one group and satisfying relationship (1.3) another group of conditions imposed on the operator A and ensuring that the method can be used to find and construct periodic solutions of equations (1.1). The character of these conditions depends essentially on the properties of both the operator A and its "extension", i. e. on the properties of the operator $\tilde{A}$ and the operator B which it generates.

It should be noted that as a rule in particular realizations of the method analytic means are used to construct the limiting function u^0, and numerical methods to identify it with the periodic solutions of equation (1.1).

Having outlined the idea of the method by the example of the operator equation, we proceed to present it as applied to

ordinary differential equations. The combination of conditions ensuring that the method is applied to differential equations separates a class of quite definite systems which have been rather conventionally called *T-systems* [162, 163].

2. CONDITIONS SEPARATING A T-SYSTEM. THE CONCEPT OF A Δ-CONSTANT

Let $x = (x_1, x_2, \ldots, x_n)$ be a point in an n-dimensional Euclidean space E_n and D a closed bounded domain in E_n and Γ its boundary.

For the point x we denote by $|x|$ the vector with the coordinates $(|x_1|, |x_2|, \ldots, |x_n|)$ and by $|x|$ the *octahedral norm* equal to $\sum_{i=1}^{n} |x_i|$.

We make it a convention to write vectors and matrices with numerical nonnegative components in bold type and to understand the inequalities between them component-wise (for example, for the vectors $\mathbf{K} = (K_1, K_2, \ldots, K_n)$ and $\mathbf{K}' = (K_1', K_2', \ldots, K_n')$ the inequality $\mathbf{K} \leqslant \mathbf{K}'$ means $K_i \leqslant K_i'$ for all $i = 1, 2, \ldots, n$).

By the $\mathbf{d}$-*neighbourhood* of a point $\bar{x} \in E_n$ we shall understand a set of points $x \in E_n$ for which $|x - \bar{x}| < \mathbf{d}$.

Let the function $f(t, x) = (f_1(t, x), f_2(t, x), \ldots, f_n(t, x))$ be defined and continuous in the domain

$$(t, x) \in R^1 \times D = (-\infty, \infty) \times D \tag{2.1}$$

and periodic in t of period T with values in E_n. We shall assume that the function $|f(t, x)| = (|f_1(t, x)|, |f_2(t, x)|, \ldots, |f_n(t, x)|)$ is bounded by the vector $\mathbf{M}$

$$|f(t, x)| \leqslant \mathbf{M} \tag{2.2}$$

and satisfies a Lipschitz condition with the matrix $\mathbf{K}$

$$|f(t, x') - f(t, x'')| \leqslant \mathbf{K} |x' - x''| \tag{2.3}$$

for all $t \in R^1$ and $x, x', x'' \in D$.

We connect with the function $f(t, x)$ and domain D and denote by D_f a set of points of E_n contained in D together with its $(T/2)\mathbf{M}$-neighbourhood.

Definition 1. *A system of differential equations*

$$\frac{dx}{dt} = f(t, x) \tag{2.4}$$

the right-hand side of which is defined, continuous and periodic in t of period T in domain (2.1) *will be called a T-system in the domain D if*

(1) *the set* D_f *is nonempty*

$$D_f \neq \varnothing \tag{2.5}$$

(2) *the greatest eigenvalue* $\lambda_{\max}$ *of the matrix* $\mathbf{Q} = (T/\pi)\mathbf{K}$ *does not exceed unity*

$$\lambda_{\max} < 1 \tag{2.6}$$

From the Perron theorem [8] it is known that the largest eigenvalue of the matrix $(T/\pi)\mathbf{K}$ in virtue of the nonnegativeness of its elements $(T/\pi)\mathbf{K}_{ij}$ is real, nonnegative and estimated from above by a value equal to

$$\frac{T}{\pi}\min\left\{\max_i \sum_{j=1}^{n} K_{ij},\ \max_j \sum_{i=1}^{n} K_{ij}\right\} \tag{2.7}$$

This, in particular, leads to the fact that if inequality (2.3) is given by the vector $\mathbf{K}$ in the form

$$|f(t, x') - f(t, x'')| \leqslant \mathbf{K}|x' - x''| \tag{2.8}$$

or by the scalar K in the form

$$|f_i(t, x') - f_i(t, x'')| \leqslant K|x' - x''|,\ i = 1, 2, \ldots, n \tag{2.9}$$

then condition (2.6) holds for

$$q_1 = \frac{|\mathbf{K}|T}{\pi} < 1 \quad \text{or} \quad q_2 = \frac{KnT}{\pi} < 1 \tag{2.10}$$

respectively.

Because $\mathbf{M}$ and $\mathbf{K}$ depend essentially on the domain D, it may prove that system (2.4) is a T-system with respect to some subdomains of D, while with respect to others it is not. We shall give some examples to illustrate this situation.

EXAMPLE 2.1. Let the function $f(t, x)$ be defined so as to satisfy conditions (2.2) and (2.3) for all $(t, x) \in R^1 \times E_n$. Then, if condition (2.6) holds for the matrix $\mathbf{K}$, system (2.4) is a T-system in any closed, bounded domain D containing an n-dimensional rectangle the length of whose sides is not less than the corresponding components of the vector $(T/2)\mathbf{M}$.

EXAMPLE 2.2. Let the function $f(t, x)$ be linear

$$f(t, x) = P(t)x + q(t) \tag{2.11}$$

We put

$$\mathbf{P} = \left\{ \max_{t\in[0,T]} |P_{ij}(t)| \right\}$$

$$\mathbf{q} = \left(\max_{t\in[0,T]} |q_1(t)|, \ldots, \max_{t\in[0,T]} |q_n(t)| \right) \tag{2.12}$$

System (2.4) can be a T-system only when the greatest eigenvalue of the matrix $(T/\pi)\mathbf{P}$ does not exceed one and it is a T-system in the domain

$$|x| \leqslant \mathbf{d} \tag{2.13}$$

when this eigenvalue is smaller than $2/\pi$ and

$$\mathbf{d} = \left(E - \frac{T}{2}\mathbf{P}\right)^{-1} \left(\frac{T}{2}\mathbf{q} + \mathbf{d}_1\right) \tag{2.14}$$

where $\mathbf{d}_1$ is an arbitrary positive vector.

EXAMPLE 2.3. Let the system of equations

$$\frac{dx_1}{dt} = x_2, \quad \frac{dx_2}{dt} = -x_1 - \sin 10t x_2^s + 1.5\cos 10t \tag{2.15}$$

be defined in the domain

$$D: \quad |x_1| \leqslant a, \quad |x_2| \leqslant 1 \tag{2.16}$$

Then for $s \geqslant 0$

$$\mathbf{M} = (1,\ a + 2.5), \qquad \mathbf{K} = \begin{pmatrix} 0 & 1 \\ 1 & s \end{pmatrix}, \qquad T = \frac{\pi}{5}$$

$$D_f: \quad |x_1| \leqslant a - \frac{\pi}{10}, \quad |x_2| \leqslant 1 - \frac{\pi}{10}(a + 2.5)$$

$$\lambda_{\max} = \frac{\pi}{15} \frac{s + (s^2 + 4)^{1/2}}{2}$$

Conditions (2.5) and (2.6) hold when

$$\frac{\pi}{10} \leqslant a \leqslant \frac{10}{\pi} - \frac{5}{2} \tag{2.17}$$

$$s \leqslant \frac{15}{\pi} - \frac{\pi}{15} \tag{2.18}$$

Thus for every a in the interval (2.17) system (2.15) determines a $(\pi/5)$-system in the domain (2.16) if s satisfies inequality (2.18).

In what follows we shall investigate periodic solutions of T-systems. Besides (2.4) we shall also consider the system of equations

$$\frac{dx}{dt} = f(t, x) - \mu \tag{2.19}$$

where $\mu = (\mu_1, \ldots, \mu_n)$ is a parameter.

Definition 2. *The value of the parameter* μ *at which the solution of system* (2.19) *taking for* $t = \tau$ *the value* $x = x_0$ *is periodic of* T *and such a* μ *is unique will be called a* Δ*-constant at the point* (τ, x_0) *with respect to the system of equations*

$$\frac{dx}{dt} = f(t, x) \tag{2.20}$$

It is evident that the points (τ, x_0), for which the Δ-constant is zero, determine the initial values of the periodic solutions of system (2.19).

3. SUBSIDIARY STATEMENTS

We give some statements which we shall need below.

Lemma 3.1. *Let* $f(t)$ *be a function continuous in the interval*

$$\tau \leqslant t \leqslant \tau + T \tag{3.1}$$

Then

$$\left| \int_\tau^t \left(f(t) - \frac{1}{T} \int_\tau^{\tau+T} f(s)\, ds \right) dt \right| \leqslant 2(t-\tau)\left(1 - \frac{t-\tau}{T}\right) \max_{t \in [\tau, \tau+T]} |f(t)|$$

$$= \alpha_1(t) \max_{t \in [\tau, \tau+T]} |f(t)| \tag{3.2}$$

for all t *on the interval* (3.1).

The proof follows directly from the estimate

$$\left|\int_\tau^t \left(f(t) - \frac{1}{T}\int_\tau^{\tau+T} f(s)\,ds\right)dt\right|$$

$$\leqslant \left|\left(1 - \frac{t-\tau}{T}\right)\int_\tau^t f(s)\,ds\right| + \left|\frac{t-\tau}{T}\int_t^{\tau+T} f(s)\,ds\right|$$

Lemma 3.2. *Given the sequence of functions* $\alpha_0 = 1$, $\alpha_1(t)$, $\alpha_2(t)$, $\alpha_m(t)$, ... *defined by the recurrence relation*

$$\alpha_{m+1}(t) = \left(1 - \frac{t-\tau}{T}\right)\int_\tau^t \alpha_m(s)\,ds + \frac{t-\tau}{T}\int_t^{\tau+T} \alpha_m(s)\,ds \qquad (3.3)$$

$$m = 0,\ 1,\ 2, \ldots$$

there exist positive constants

$$q_0 = 1,\ q_1 = \frac{1}{3},\ q_2 = \frac{1}{10},\ \ldots$$

$$\frac{1}{5^m} < q_m \leqslant \frac{1}{10^{[m/2]}3^{2(m/2)}},\ \ldots \qquad (3.4)$$

where $[m/2]$ *and* $(m/2)$ *are the integral and fractional parts of the number* $m/2$, *respectively, such that*

$$\alpha_{m+1}(t) \leqslant T^m q_m 2(t-\tau)\left(1 - \frac{t-\tau}{T}\right) = T^m q_m \alpha_1(t) \qquad (3.5)$$

for all t *in the interval* (3.1).

Proof. Direct calculations show that

$$\alpha_1(t) = 2(t-\tau)\left(1 - \frac{t-\tau}{T}\right)$$

$$\alpha_2(t) = \alpha_1(t)\left(\frac{T}{6} + \frac{\alpha_1(t)}{3}\right) \leqslant \frac{T}{3}\alpha_1(t)$$

$$\alpha_3(t) = \alpha_1(t)\left(\frac{T^2}{20} + \frac{T}{15}\alpha_1(t) + \frac{1}{15}\alpha_1^2(t)\right) \leqslant \frac{T^2}{10}\alpha_1(t) \qquad (3.6)$$

for all $t \in [\tau,\ \tau + T]$.

Thus when $m = 0, 1, 2$ inequality (3.5) holds. From relations (3.3) and (3.6) it follows that when $t \in [\tau, \tau + T]$

$$\alpha_4(t) \leqslant \frac{T^2}{10}\alpha_2(t) \leqslant \frac{T^3}{30}\alpha_1(t)$$

$$\alpha_m(t) \leqslant \frac{T^2}{10}\alpha_{m-2}(t), \qquad m \geqslant 3 \tag{3.7}$$

We now show that inequalities (3.7) imply the estimate

$$\alpha_m(t) \leqslant \frac{T^{m-1}}{10^{[m-1/2]}3^{2(m-1/2)}}\alpha_1(t), \qquad m = 2, 3, \ldots \tag{3.8}$$

Indeed, for $m = 2$ inequality (3.8) holds. We then assume that it is valid for $m \leqslant l$. Setting $m = l + 1$, on taking (3.7) and (3.8) into account we find that

$$\alpha_{l+1}(t) \leqslant \frac{T^2}{10}\alpha_{l-1}(t)$$

$$\leqslant \frac{T^2}{10}\,\frac{T^{l-2}\alpha_1(t)}{10^{[l/2-1]}3^{2(l/2-1)}} = \frac{T^l\alpha_1(t)}{10^{[l/2]}3^{2(l/2)}} \tag{3.9}$$

for all $t \in [\tau, \tau + T]$.

We conclude by induction that inequality (3.8) is valid for all $m \geqslant 2$. This leads to the estimate

$$\alpha_{m+1}(t) \leqslant T^m q_m \alpha_1(t) \tag{3.10}$$

for all $m \geqslant 0$ and $\tau \leqslant t \leqslant \tau + T$ as soon as

$$q_m \leqslant \frac{1}{10^{[m/2]}3^{2(m/2)}}, \quad m = 2, 3, \ldots \tag{3.11}$$

Now we obtain the estimate for q_m from below. By formulas (3.6),

$$\alpha_3(t) \geqslant \frac{T\alpha_1(t)}{5}\left(\frac{T}{4} + \frac{\alpha_1(t)}{3}\right) = \frac{T\alpha_1(t)}{5}\left(\frac{T}{6} + \frac{\alpha_1(t)}{3} + \frac{T}{12}\right)$$

$$= \frac{T}{5}\alpha_2(t) + \frac{T^2}{5 \times 12}\alpha_1(t)$$

hence

$$\alpha_4(t) \geqslant \frac{T}{5}\alpha_3(t) + \frac{T^2}{60}\alpha_2(t)$$

$$\geqslant \frac{T^2}{25}\alpha_2(t) + \frac{T^2}{60}\alpha_2(t) = \frac{17\,T^2}{300}\alpha_2(t) \geqslant \frac{T^2}{18}\alpha_2(t)$$

We conclude by induction that

$$\alpha_m(t) \geqslant \frac{T^2}{18}\alpha_{m-2}(t), \qquad m = 4,\ 5,\ldots \tag{3.12}$$

Just as the last inequality of (3.7) yielded estimate (3.8), inequality (3.12) yields the estimate

$$\alpha_m(t) \geqslant \frac{T^{m-2}}{18^{\left[\frac{m}{2}-1\right]}3^{2\left(\frac{m}{2}-1\right)}}\alpha_2(t)$$

$$= \frac{T^{m-2}\alpha_1(t)}{18^{\left[\frac{m}{2}-1\right]}3^{2\left(\frac{m}{2}-1\right)}}\left(\frac{T}{6} + \frac{\alpha_1(t)}{3}\right)$$

$$\geqslant \frac{T^{m-1}\alpha_1(t)}{6 \times 18^{\left[\frac{m}{2}-1\right]}3^{2\left(\frac{m}{2}-1\right)}}$$

for $m \geqslant 4$. Then for q_m we have the inequality

$$T^m q^m \alpha_1(t) \geqslant \frac{T^m}{6 \times 18^{\left[\frac{m+1}{2}-1\right]}3^{2\left(\frac{m+1}{2}-1\right)}}\alpha_1(t) \tag{3.13}$$

from which the estimate

$$q_m \geqslant \frac{1}{5^m} \tag{3.14}$$

follows, which completes the proof of the lemma.

4. THE MAIN LEMMA

With each fixed $x \in D$ for the function $f(t,\ x) = (f_1(t,\ x), f_2(t,\ x), \ldots, f_n(t,\ x))$ defined in the domain (2.1), we shall denote by $|f(t,\ x)|_0$ the vector with coordinates $(\sup_t |f_1(t,\ x)|, \ldots, \sup_t$

$|f_n(t, x)|$); and by $\overline{f(t, x)}$ the time average over the interval [0, T]:

$$\overline{f(t, x)} = \frac{1}{T}\int_0^T f(s, x(s))\, ds$$

Investigations of periodic solutions of T-systems make essential use of the statements and estimates given below.

The Main Lemma. *Let the function $f(t, x)$ be defined in the domain* (2.1), *continuous in t, x, periodic in t of period T and satisfy inequalities* (2.2), (2.3), (2.5) *and* (2.6). *Then the sequence of functions*

$$x_m(t, \tau, x_0) = x_0 + \int_\tau^t (f(t, x_{m-1}(t, \tau, x_0)) - \overline{f(t, x_{m-1}(t, \tau, x_0))})\, dt \tag{4.1}$$

periodic in t of period T, converges, when $m \to \infty$, uniformly with respect to

$$(t, \tau, x_0) \in (-\infty, \infty) \times (-\infty, \infty) \times D_f \tag{4.2}$$

to the function $x^0(t, \tau, x_0)$ defined in domain (4.2), *periodic in t, τ of period T and satisfying the system of equations*

$$x(t, \tau, x_0) = x_0 + \int_\tau^t (f(t, x(t, \tau, x_0)) - \overline{f(t, x(t, \tau, x_0))})\, dt \tag{4.3}$$

with

$$|x^0(t, \tau, x_0) - x_m(t, \tau, x_0)|$$

$$\leqslant \alpha_1(t)\mathbf{Q}^m\left((E - \mathbf{Q})^{-1} + \sum_{j=1}^{3} \delta_{jm}\mathbf{B}_j\right)\mathbf{M} \tag{4.4}$$

for all $m \geqslant 1$ and $t \in R^1$, where

$$\delta_{jm} = \begin{cases} 0 & \text{when } j \neq m \\ 1 & \text{when } j = m \end{cases}$$

$$\mathbf{B}_1 = \frac{1}{30\pi^2}((10\pi^3 - 30\pi^2)E + (3\pi^3 - 30\pi)T\mathbf{K} + (\pi^3 - 30)T^2\mathbf{K}^2)$$

$$\mathbf{B}_2 = \frac{1}{30\pi}((3\pi^3 - 30\pi)E + (\pi^3 - 30)T\mathbf{K})$$

$$\mathbf{B}_3 = \frac{\pi^3 - 30}{30}E \tag{4.5}$$

Proof. Consider the sequence of functions $x_1(t, \tau, x_0), x_2(t, \tau, x_0), \ldots$ $\ldots, x_m(t, \tau, x_0), \ldots$ defined by the recurrence relation (4.1). Each of the functions of the sequence is periodic in t of period T. Moreover, when $x_0 \in D$, by Lemma 3.1

$$|x_1 - x_0| \leqslant 2(t-\tau)\left(1 - \frac{t-\tau}{T}\right)\mathbf{M} \leqslant \frac{T}{2}\mathbf{M}$$

It follows that $x_1(t, \tau, x_0) \in D$ when $x_0 \in D_f$. If, however, $x_{m-1}(t, \tau, x_0) \in D$, then in virtue of Lemma 3.1 it follows from (4.1) that

$$|x_m(t, \tau, x_0) - x_0| \leqslant \frac{T}{2}\mathbf{M}$$

i. e. $x_m(t, \tau, x_0) \in D$ when $x_0 \in D_f$. We conclude by induction that for all $m \geqslant 0$, any $(t, \tau) \in (-\infty, \infty) \times (-\infty, \infty)$ and every $x_0 \in D_f$ the functions $x_m(t, \tau, x_0)$ belong to the set D.

From (4.4) we now find that

$$\begin{aligned}
&|x_{m+1}(t, \tau, x_0) - x_m(t, \tau, x_0)| \\
&\leqslant \left(1 - \frac{t-\tau}{T}\right)\int_\tau^t \mathbf{K}|x_m(t, \tau, x_0) - x_{m-1}(t, \tau, x_0)|\,dt \\
&\quad + \frac{t-\tau}{T}\int_t^{\tau+T} \mathbf{K}|x_m(t, \tau, x_0) - x_{m-1}(t, \tau, x_0)|\,dt
\end{aligned} \tag{4.6}$$

for all $m \geqslant 1$ and all t in the interval $\tau \leqslant t \leqslant \tau + T$.

We introduce the notation

$$r_{m+1}(t) = |x_{m+1}(t, \tau, x_0) - x_m(t, \tau, x_0)| \tag{4.7}$$

and rewrite inequality (4.6) in the form

$$r_{m+1}(t) \leqslant \mathbf{K}\left(\left(1 - \frac{t-\tau}{T}\right)\int_\tau^t r_m(s)\,ds + \frac{t-\tau}{T}\int_t^{\tau+T} r_m(s)\,ds\right) \tag{4.8}$$

Since $r_1(t) \leqslant \mathbf{M}\alpha_1(t)$, $t \in [\tau, \tau + T]$, from relation (4.8) we have

$$r_2(t) \leqslant \mathbf{KM}\left(\left(1 - \frac{t-\tau}{T}\right)\int_{\tau}^{t} \alpha_1(s)\,ds + \frac{t-\tau}{T}\int_{t}^{\tau+T} \alpha_1(s)\,ds\right)$$

$$= \mathbf{KM}\alpha_2(t)$$

$$r_3(t) \leqslant \mathbf{K}^2\mathbf{M}\alpha_3(t)$$

. . . . . .

$$r_{m+1}(t) \leqslant \mathbf{K}^m\mathbf{M}\alpha_{m+1}(t), \quad t \in [\tau, \tau + T], \quad m = 0, 1, 2, \ldots \tag{4.9}$$

Since $\alpha_{m+1}(t) \leqslant T^m q_m \alpha_1(t)$ in virtue of Lemma 3.2, from estimate (4.9) we find that

$$r_{m+1}(t) \leqslant T^m q_m \mathbf{K}^m \mathbf{M}\alpha_1(t), \quad m \geqslant 0, \quad t \in [\tau, \tau + T] \tag{4.10}$$

From (4.10) we conclude that for any $k \geqslant 1$ we have the inequality

$$|x_{m+k}(t, \tau, x_0) - x_m(t, \tau, x_0)| \leqslant \sum_{i=0}^{k-1} (T\mathbf{K})^{m+i} q_{m+i} \mathbf{M}\alpha_1(t)$$

for $t \in [\tau, \tau + T]$, or, taking account of estimate (3.11) for q_m, the inequality

$$|x_{m+k}(t,\tau,x_0) - x_m(t,\tau,x_0)| \leqslant \sum_{i=0}^{k-1} (T\mathbf{K})^{m+i} \frac{\mathbf{M}\alpha_1(t)}{10^{\left[\frac{m+i}{2}\right]} 3^{2\left(\frac{m+i}{2}\right)}} \tag{4.11}$$

Since

$$\frac{1}{10^{\left[\frac{m+i}{2}\right]} 3^{2\left(\frac{m+i}{2}\right)}} \leqslant \frac{1}{\pi^{m+i}} \quad \text{when} \quad m + i \geqslant 4 \tag{4.12}$$

we see that (4.11) yields the estimate

$$|x_{m+k}(t, \tau, x_0) - x_m(t, \tau, x_0)| \leqslant \sum_{i=0}^{k-1} \left(\frac{T\mathbf{K}}{\pi}\right)^{m+i} \mathbf{M}\alpha_1(t) \tag{4.13}$$

for all $m \geqslant 4$ and $t \in [\tau, \tau + T]$.

The eigenvalues of the matrix $\mathbf{Q} = (T/\pi)\mathbf{K}$ are assumed to lie within the circle of a unit radius, which implies that

$$\sum_{i=0}^{k-1} \left(\frac{T}{\pi}\mathbf{K}\right)^{m+i} \leqslant \mathbf{Q}^m \sum_{i=0}^{\infty} \mathbf{Q}^i = \mathbf{Q}^m (E - \mathbf{Q})^{-1} \tag{4.14}$$

$$\lim_{m \to \infty} \mathbf{Q}^m = 0 \tag{4.15}$$

Relations (4.13) to (4.15) prove the uniform convergence of the function $x_m(t, \tau, x_0)$ in $(t, \tau, x_0) \in R^1 \times R^1 \times D_f$ as $m \to \infty$

$$\lim_{m \to \infty} x_m(t, \tau, x_0) = x^0(t, \tau, x_0) \tag{4.16}$$

Since the functions $x_m(t, \tau, x_0)$ are periodic in t, τ of period T, the limiting function is also periodic in t, τ of period T.

Inequality (4.13) gives for the difference $|x^0(t, \tau, x_0) - x_m(t, \tau, x_0)|$ the estimate

$$|x^0(t, \tau, x_0) - x_m(t, \tau, x_0)| \leqslant \frac{T}{2}\mathbf{Q}^m (E - \mathbf{Q})^{-1}\mathbf{M} \tag{4.17}$$

for all $m \geqslant 4$ and $t \in R^1$.

For the differences $|x^0(t, \tau, x_0) - x_m(t, \tau, x_0)|$, where $m = 0, 1, 2, 3$, it is easy to obtain the estimates

$$|x^0(t,\tau,x_0) - x_0| \leqslant \frac{T}{2}\mathbf{M} \tag{4.18}$$

$$|x^0(t,\tau,x_0) - x_1(t,\tau,x_0)| \leqslant \alpha_1(t)\mathbf{Q}((E - \mathbf{Q})^{-1} + \mathbf{B}_1)\mathbf{M} \tag{4.19}$$

$$|x^0(t,\tau,x_0) - x_2(t,\tau,x_0)| \leqslant \alpha_1(t)\mathbf{Q}^2((E - \mathbf{Q})^{-1} + \mathbf{B}_2)\mathbf{M} \tag{4.20}$$

$$|x^0(t,\tau,x_0) - x_3(t,\tau,x_0)| \leqslant \alpha_1(t)\mathbf{Q}^3((E - \mathbf{Q})^{-1} + \mathbf{B}_3)\mathbf{M} \tag{4.21}$$

where $\alpha_1(t) = 2(t - \tau)(1 - (t - \tau)/T) \leqslant T/2$.

Using relation (4.16) and proceeding in equation (4.1) to the limit, when $m \to \infty$, convinces us that the function $x^0(t, \tau, x_0)$ is the periodic solution of equation (4.3).

Thus the proof of the main lemma is complete.

Note that if a Lipschitz condition for the function $f(t, x)$ is given by inequalities (2.8) and (2.9), estimate (4.17) takes on a simpler

form:

$$|x^0(t, \tau, x_0) - x_m(t, \tau, x_0)|$$

$$\leqslant \alpha_1(t)\frac{|\mathbf{M}|}{|\mathbf{K}|}\left(\frac{1}{1-q_1} + \sum_{j=1}^{3}\delta_{jm}\mathbf{B}_j\right)q_1^m\mathbf{K} \tag{4.22}$$

or

$$|x_i^0(t, \tau, x_0) - x_{mi}(t, \tau, x_0)|$$

$$\leqslant \alpha_1(t)\frac{|\mathbf{M}|}{n}\left(\frac{1}{1-q_2} + \sum_{j=1}^{3}\delta_{jm}\mathbf{B}_j\right)q_2^m \tag{4.23}$$

for all $m \geqslant 1$ and $t \in R^1$.

5. PROPERTIES OF Δ-CONSTANTS FOR POINTS OF THE SET $R^1 \times D_f$

The main lemma gives an easy solution to the problem of the existence of Δ-constants for the points of the set $R^1 \times D_f$. The following theorem holds for a T-system.

Theorem 5.1. *Any point* $(\tau, x_0) \in R^1 \times D_f$ *has a* Δ*-constant with respect to a* T*-system given in* D.

Proof. Since for a T-system the right-hand side, i. e. the function $f(t, x)$, satisfies the conditions of the main lemma, the sequence of the periodic functions

$$x_m(t, \tau, x_0)$$

$$= x_0 + \int_\tau^t \left(f(s, x_{m-1}(s, \tau, x_0)) - \overline{f(t, x_{m-1}(t, \tau, x_0))}\right)ds \tag{5.1}$$

converges uniformly for any $(\tau, x_0) \in R^1 \times D_f$ to the periodic function $x^0(t, \tau, x_0)$. Putting

$$\Delta(\tau, x_0) = \overline{f(t, x^0(t, \tau, x_0))} \tag{5.2}$$

we show that $\Delta(\tau, x_0)$ is a Δ-constant at the point τ, x_0 with respect to the T-system given in D. It follows from the main lemma that when (5.2) is satisfied $x^0(t, \tau, x_0)$ is a periodic solution of period T of the system of equations

$$\frac{dx}{dt} = f(t, x) - \Delta \tag{5.3}$$

passing at $t = \tau$ through the point x_0. Therefore, to prove the theorem it remains to be shown that for any value of parameter Δ different from (5.2) the solutions of system (5.3) passing at $t = \tau$ through the point x_0 are not periodic of period T.

To prove this, suppose the contrary. There would then exist constants u_1, u_2, $u_1 \neq u_2$, such that the solutions $x(t, \tau, x_0, \Delta)$ of the system of equations (5.3) passing at $t = \tau$ through the point x_0 are periodic of period T when $\Delta = u_1$ and $\Delta = u_2$. For the difference $x(t, \tau, x_0, u_2) - x(t, \tau, x_0, u_1)$ we would then have the identity

$$x(t, u_2) - x(t, u_1) = \int_\tau^t (f(s, x(s, u_2)) - \overline{f(s, x(s, u_2))}$$

$$- \int_\tau^t (f(s, x(s, u_1)) + \overline{f(s, x(s, u_1))})\, ds \tag{5.4}$$

from which we find

$$|x(t, u_2) - x(t, u_1)|$$

$$\leqslant \mathbf{K}\left(\left(1 - \frac{t-\tau}{T}\right)\int_\tau^t |x(s, u_2) - x(s, u_1)|\, ds \right.$$

$$\left. + \frac{t-\tau}{T} \int_t^{\tau+T} |x(s, u_2) - x(s, u_1)|\, ds\right) \tag{5.5}$$

Putting

$$|x(t, u_2) - x(t, u_1)| = r(t) \tag{5.6}$$

we find from (5.5)

$$r(t) \leqslant \alpha_m(t)\, \mathbf{K}^m |r(t)|_0, \quad m = 0, 1, 2, \ldots \tag{5.7}$$

where $\alpha_0 = 1$, $\alpha_1(t), \ldots, \alpha_m(t), \ldots$ are positive functions defined by relation (3.3).

By Lemma 3.2 we have

$$\alpha_{m+1}(t) \leqslant \frac{T}{2} \frac{T^m}{10^{\left[\frac{m}{2}\right]} 3^{2\left[\frac{m}{2}\right]}} \tag{5.8}$$

so that inequality (5.7) yields

$$|r(t)|_0 \leqslant \frac{T}{2}\left(\frac{T}{\pi}\mathbf{K}\right)^m |r(t)|_0, \qquad m \geqslant 4 \tag{5.9}$$

Under the assumption that the eigenvalues of the matrix $\mathbf{Q}$ lie in the circle of a unit radius inequality (5.9) is possible only when

$$|r(t)|_0 = 0 \tag{5.10}$$

and hence when $u_1 = u_2$. The contradiction proves the theorem.

If the set D_f does not degenerate to a point, a Δ-constant of the T-system may be considered as the function $\Delta = \Delta(\tau, x_0)$ given on the set $R^1 \times D_f$. The properties of this function are defined by

Theorem 5.2. *A Δ-constant of points $(\tau, x_0) \in R^1 \times D_f$ with respect to a T-system given in D is a continuous function $\Delta = \Delta(\tau, x_0)$ periodic in τ of period T and satisfying the inequalities*

$$|\Delta(\tau, x_0) - \Delta(t, x_0')|$$

$$\leqslant \mathbf{K}\left(E + \frac{T}{3}\mathbf{K}\left(E - \frac{T}{\pi}\mathbf{K}\right)^{-1} + \mathbf{B}\right)|x_0 - x_0'|$$

$$|\Delta| \leqslant \mathbf{M} \tag{5.11}$$

for all τ, x_0, x_0' of the domain

$$(\tau, x_0) \in R^1 \times D_f, \qquad x_0' \in D_f \tag{5.12}$$

where

$$\mathbf{B} = \frac{1}{90\pi^3}((10\pi^3 - 30\pi^2)T^2\mathbf{K}^2$$

$$+ (3\pi^3 - 30\pi)T^3\mathbf{K}^3 + (\pi^3 - 30)T^4\mathbf{K}^4) \tag{5.12'}$$

Proof. For any point τ, x_0 of the domain (5.12) its Δ-constant with respect to a T-system given in D is defined by formula (5.2) where $x^0(t, \tau, x_0)$ is the limit of a sequence of periodic functions $x_m(t, \tau, x_0)$ given according to (5.1). From the properties of the function $x^0(t, \tau, x_0)$ established by the main lemma of Section 4 it follows that the function $\Delta(\tau, x_0)$ is continuous, periodic and bounded when (τ, x_0) changes in the domain (5.12). Equality (5.2) yields

$$|\Delta(\tau, x_0) - \Delta(\tau, x_0')|$$

$$\leqslant \frac{1}{T}\mathbf{K}\int_0^T |x^0(s, \tau, x_0) - x^0(s, \tau, x_0')|\,ds \tag{5.13}$$

for any x_0, $x_0' \in D_f$.

Taking into account that $x^0(t, \tau, x_0)$ satisfies equation (4.3), we find that

$$|x^0(t, \tau, x_0) - x^0(t, \tau, x_0')|$$

$$\leqslant |x_0 - x_0'| + \mathbf{K}\left(\left(1 - \frac{t-\tau}{T}\right)\int_\tau^t |x^0(s, \tau, x_0) - x^0(s, \tau, x_0')|\,ds\right.$$

$$\left. + \frac{t-\tau}{T}\int_t^{\tau+T} |x^0(s,\tau,x_0) - x^0(s,\tau,x_0')|\,ds\right) \tag{5.14}$$

Putting $|x^0(t, \tau, x_0) - x^0(t, \tau, x_0')| = r(t)$ and solving inequality (5.14), we obtain

$$r(t) \leqslant (E + \mathbf{K}\alpha_1(t) + \mathbf{K}^2\alpha_2(t) + \ldots + \mathbf{K}^m\alpha_m(t))\,|x_0 - x_0'|$$

$$+ \mathbf{K}^{m+1}\alpha_{m+1}(t)\,|r(t)|_0 \tag{5.15}$$

where $\alpha_1(t), \ldots, \alpha_m(t), \ldots$ are positive functions satisfying relation (3.3).

By statements of Lemma 3.2 it follows from inequality (5.15) that

$$r(t) \leqslant \left(E + \sum_{m=0}^{\infty} \frac{T^m\mathbf{K}^m\mathbf{K}}{10^{[m/2]}3^{2(m/2)}}\right)|x_0' - x_0''| \tag{5.16}$$

Since $\int_0^T \alpha_1(t)\,dt = T^2/3$, from (5.13) and (5.16) we find that

$$|\Delta(\tau, x_0) - \Delta(\tau, x_0)|$$

$$\leqslant \mathbf{K}\left(E + \frac{T}{3}\mathbf{K}(E - \mathbf{Q})^{-1} + \frac{1}{90\pi^3}((10\pi^3 - 30\pi^2)\,T^2\mathbf{K}^2\right.$$

$$\left. + (3\pi^3 - 30\pi)\,T^3\mathbf{K}^3 + (\pi^3 - 30)\,T^4\mathbf{K}^4)\right)|x_0 - x_0'| \tag{5.17}$$

for all

$$x_0,\ x_0' \in D_f \tag{5.18}$$

The last inequality completes the proof of the theorem.

6. PERIODIC PROBLEM OF CONTROL

The existence, finding and properties of a Δ-constant will play the principal part in the construction of periodic solutions of T-systems. They are of interest in themselves, too, and especially in connection with the following problem of control.

It is required for a system whose motion is described by equation (2.19) with the right-hand side periodic in t of period T and the constant control $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ that a control should be chosen so that a trajectory passing at a given time $t = \tau$ through a given point $x = x_0$ shall be made periodic of period T. It is clear that the solution to this problem, which it is natural to call a *periodic problem of control*, is given by a control equal to the Δ-constant at the point (τ, x_0) with respect to system (2.20).

If system (2.20) is a T-system in the domain D and the point x_0 belongs to the set D_f, then, as a consequence of Theorem 5.1, such a control exists, is unique and can be found approximately using the formula

$$\mu = \mu_m = \overline{f(t, x_m(t, \tau, x_0))} \tag{6.1}$$

where the function $x_m(t, \tau, x_0)$ is defined by the recurrence relation (5.1). Moreover, Theorem 5.2 states that this control is stable in the sense that the small change of the point x_0 through which the periodic trajectory must pass requires a small change in the control μ.

It is easy to see that the stability of control is not changed by a small variation in time τ.

As an example, consider a periodic problem of control of a linear T-system.

Find the control $\mu = \mu(x_0)$ for the solution of the system of equations

$$\frac{dx}{dt} = P(t)x - \mu, \qquad P(t + T) = P(t) \tag{6.2}$$

which at $t = 0$ takes the value $x = x_0$, to be periodic of period T.

Assuming the homogeneous system of equations (6.2) in the domain (2.13) to be a T-system, for which it is sufficient that the greatest eigenvalue of the matrix $(T/\pi)\mathbf{P}$ should be less than $2/\pi$, the control μ being sought is found as the limit of the successive approximations

$$\mu_m = \frac{1}{T}\int_0^T P(s)x_m(s)\,ds \tag{6.3}$$

where $x_m(t)$ are periodic functions defined by the relation

$$x_m(t) = x_0 + \int_0^t P(s)\, x_{m-1}(s)\, ds \tag{6.4}$$

$$m = 1, 2, 3, \ldots \text{ and } x_0(t) = x_0$$

Here for the deviation of μ from the approximate value of the control μ_m computed according to (6.3) the following estimate holds

$$|\mu - \mu_m| \leqslant \frac{1}{T}\mathbf{P}\int_0^T |x_m(s) - x^0(s)|\, ds$$

$$\leqslant \frac{T}{2}\mathbf{P}\mathbf{Q}^m\left((E - \mathbf{Q})^{-1} + \sum_{j=1}^{3} \delta_{jm}\mathbf{B}_j\right)\mathbf{M} \tag{6.5}$$

where $\mathbf{M} = \mathbf{P}(E - (T/2)\mathbf{P})^{-1}|x_0|$, $\mathbf{P} = \{\max_{t\in[0,T]} |P_{ij}(t)|\}$, $m \geqslant 1$.

7. THE EXISTENCE OF PERIODIC SOLUTIONS FOR T-SYSTEMS

As has already been noted before, points (τ, x_0), for which the Δ-constant is zero, determine the initial values of the periodic solutions of the systems under consideration.

Theorem 5.1 readily yields the statement that the periodic solutions of T-system which pass through the points of the domain D_f are the limits of the sequence of periodic functions (4.1), i. e.

Corollary 7.1. *The solution $x = x(t)$ of the system of equations*

$$\frac{dx}{dt} = f(t, x)$$

which is a T-system in the domain, for which $x(\tau) = x_0 \in D_f$, will be periodic of period T if and only if the Δ-constant at the point (τ, x_0) is zero, the following equality holding

$$x(t) = x^0(t, \tau, x_0) \tag{7.1}$$

where $x^0(t, \tau, x_0)$ is the limit of the sequence of periodic functions (4.1), and estimate (4.4) being valid for the difference $|x(t) - x_m(t, \tau, x_0)|$.

Proof. Let $x = x(t)$ be the periodic solution of period T of a T-system for which $x(\tau) = x_0$, $x_0 \in D_f$. Then the solution $x(t, \tau, x_0, \Delta)$

of the system

$$\frac{dx}{dt} = f(t, x) - \Delta \tag{7.2}$$

passing at $t = \tau$ through the point x_0 is periodic of period T when $\Delta = 0$ because it equals the function $x(t)$ when $\Delta = 0$. System (7.2) when $\Delta = 0$ is a T-system in the domain D. Consequently, by Theorem 5.1, the point (τ, x_0) has a Δ-constant with respect to this T-system. By the definition of a Δ-constant the solution $x(t, \tau, x_0, \Delta)$ of system (7.2) is periodic of period T only for one value of Δ. From this we conclude that the Δ-constant is zero at the point (τ, x_0) and

$$x(t) = x^0(t, \tau, x_0) \tag{7.3}$$

where $x^0(t, \tau, x_0)$ is the limit of the sequence of periodic functions (4.1). Equality (7.3) leads to estimate (4.4).

The problem of the existence and construction of periodic solutions for a T-system is, according to the above corollary, equivalent to the existence of points (τ, x_0) for which the Δ-constant is zero, and to the construction of approximations $x_m(t, \tau, x_0)$.

At a fixed τ points x_0, for which $\Delta = 0$, are the singular points of the map

$$\Delta: \quad D_f \to E_n, \quad \Delta(x_0) = \overline{f(t, x^0(t, \tau, x_0))} \tag{7.4}$$

One can find the map (7.4) only approximately, say, by computing the functions

$$\Delta_m(x_0) = \overline{f(t, x_m(t, \tau, x_0))} \tag{7.5}$$

In this connection the problem arises as to how from the map (7.5) one can draw conclusions concerning the map $\Delta(x_0)$ and hence concerning the periodic solutions of a T-system.

To solve the last problem, we shall, following I. Bershtein and A. Khalanai [10], M.A. Krasnoselsky and A.I. Perov [87], use some geometrical criteria [3, Theorems 5.11 and 5.12].

We shall prove the following

Theorem 7.1. *Let the following conditions hold for the T-system*

$$\frac{dx}{dt} = f(t, x) \tag{7.6}$$

given in the domain $D \subset E_n$:

(1) *the map* (7.5), *has an isolated singular point* $\Delta_m(x^0) = 0$ *for some real* τ *and the integer* m;

(2) *the index of this point is nonzero*;

(3) *there exists a closed convex domain* D_1, *belonging to* D_f *and possessing a unique singular point* x^0, *such that on its boundary* Γ_{D_1} *the following inequality holds*

$$\inf_{x \in \Gamma_{D_1}} |\Delta_m(x)| \geqslant \frac{\pi}{3} \left| \mathbf{Q}^{m+1} \left((E - \mathbf{Q})^{-1} + \sum_{j=1}^{3} \delta_{jm} \mathbf{B}_j \right) \mathbf{M} \right| \tag{7.7}$$

where $\mathbf{Q} = (T/\pi)\mathbf{K}$, $m \geqslant 1$. *Then system* (7.6) *has a periodic solution* $x = x(t)$ *for which* $x(\tau) \in D_1$.

Proof. By definition, the index of an isolated singular point x^0 of a continuous map Δ_m is equal to the characteristic of the vector field corresponding to the map Δ_m on a sufficiently small sphere S^n with the centre at x^0. Since D_1 has no singular points different from x^0 and is homeomorphic to the unit sphere E_n (e. g. see [88]), the characteristic of the vector field of Δ_m on the sphere S^n is equal to that of the same field on Γ_{D_1}.

We shall show that the fields of Δ_m and Δ, corresponding to the maps (7.5) and (7.4) respectively, are homotopic on Γ_{D_1}. This follows from the fact that the family of the vector fields

$$V(\Theta, x_0) = \Delta_m(x_0) + \Theta(\Delta(x_0) - \Delta_m(x_0)) \tag{7.8}$$

continuous everywhere on Γ_{D_1}, which depend continuously on the parameter Θ, $0 \leqslant \Theta \leqslant 1$ and connects the fields $V(0, x_0) = \Delta_m(x_0)$ and $V(1, x_0) = \Delta(x_0)$, does not vanish anywhere on Γ_{D_1}. In fact, when $m \geqslant 1$ inequality (4.4) yields the estimate

$$|\Delta(\tau, x_0) - \Delta_m(\tau, x_0)|$$

$$\leqslant \frac{1}{T} \left| \int_0^T (f(s, x^0(s, \tau, x_0)) - f(s, x_m(s, \tau, x_0)))\, ds \right|$$

$$\leqslant \frac{1}{T} \mathbf{K}\mathbf{Q}^m \left((E - \mathbf{Q})^{-1} + \sum_{j=1}^{3} \delta_{jm} \mathbf{B}_j \right) \mathbf{M} \int_0^T \alpha_1(s)\, ds$$

$$= \frac{\pi}{3} \mathbf{Q}^{m+1} \left((E - \mathbf{Q})^{-1} + \sum_{j=1}^{3} \delta_{jm} \mathbf{B}_j \right) \mathbf{M} \tag{7.9}$$

and, consequently, the inequality

$$|V(\Theta, x_0)| \geqslant |\Delta_m(x_0)| - |\Delta_m(x_0) - \Delta(x_0)| > 0 \qquad (7.10)$$

holds in Γ_{D_1}.

It is well-known that the characteristics of homotopic vector fields on a compact space are equal to one another. Therefore, the characteristic of the field of Δ on Γ_{D_1} is equal to the index of the singular point x^0 of the field of Δ_m and hence different from zero. By Theorem 5.12 of [3], this is sufficient for the vector field of Δ to have a singular point in D_1, i. e. a point x_0^0 for which

$$\Delta(x_0^0) = 0 \qquad (7.11)$$

We have established that there exists such a point x_0^0 whose Δ-constant is zero. Then, by Corollary 7.1, the solution of the T-system passing through the point (τ, x_0^0) is periodic. Thus the theorem is proved.

Remark 7.1. The statement of Theorem 7.1 is valid for $m = 0$ as well, the only difference being that inequality (7.7) is replaced by the relation

$$\inf_{x \in \Gamma_{D_1}} \overline{|f(t, x)|} \geqslant \frac{\pi}{3} |\mathbf{QM}| \qquad (7.12)$$

The verification of condition (2) of Theorem 7.1 involves the calculation of the index of an isolated singular point. For a plane, i. e. when $E_n = E_2$, such a calculation is easy to realize [88]. For a space E_n of a dimension greater than two the calculation of the index is substantially more difficult. However, even in this case it is possible to use a formula (see [138]) or a number of criteria allowing one to conclude that its value is not zero. Thus, in particular, if Δ_m is a topological transformation of a neighbourhood D_1 of a singular point, then, as proved in [3], the index of this point is ± 1. Moreover, the index of a singular point $x = x^0$ of the map Δ_m is, if continuously differentiable, different from zero whenever the Jacobian of the function $\Delta_m(x)$ at the point $x = x^0$ is different from zero, i. e.

$$\det\left(\frac{\partial \Delta_m(x^0)}{\partial x}\right) \neq 0$$

The verification of condition (3) of Theorem 7.1 involves a suitable choice of the domain D_1. However, for a number of systems,

such as the system of standard form to be given below, this condition holds for a more or less arbitrarily chosen domain.

For the case when $E_n = E_1$, i. e. when x is a scalar, Theorem 7.1 can be strengthened by giving up the requirement that the singular point should be isolated. We have

Theorem 7.2. *Let a T-system*

$$\frac{dx}{dt} = f(t, x) \tag{7.13}$$

be defined on an interval $a \leqslant x \leqslant b$ of a straight line R^1. Assume that for a real τ and an integer $m \geqslant 1$ function (7.5) satisfies the inequalities

$$\min_{a+\frac{T}{2}M \leqslant x \leqslant b-\frac{T}{2}M} \Delta_m(x) \leqslant -\frac{\pi}{3} q^{m+1} \left(\frac{1}{1-q} + \sum_{j=1}^{2} \sigma_{jm} B_j \right) M$$

$$\max_{a+\frac{T}{2}M \leqslant x \leqslant b-\frac{T}{2}M} \Delta_m(x) \geqslant \frac{\pi}{3} q^{m+1} \left(\frac{1}{1-q} + \sum_{j=1}^{3} \delta_{jm} B_j \right) M \tag{7.14}$$

where $q = (T/\pi)K$. Then equation (7.13) has a periodic solution $x = x(t)$ for which $a + (T/2)M \leqslant x(\tau) \leqslant b - (T/2)M$.

Proof. Let x_1 and x_2 be points of the interval $[a + (T/2)M, b - (T/2)M]$ such that

$$\Delta_m(x_1) = \min \Delta_m(x), \qquad \Delta_m(x_2) = \max \Delta_m(x)$$

$$x \in \left[a + \frac{T}{2}M, \; b - \frac{T}{2}M \right]$$

Taking into account inequalities (7.9) and (7.14), we have

$$\Delta(x_1) = \Delta_m(x_1) + (\Delta(x_1) - \Delta_m(x_1)) \leqslant 0$$

$$\Delta(x_2) = \Delta_m(x_2) + (\Delta(x_2) - \Delta_m(x_2)) \geqslant 0 \tag{7.15}$$

It follows from (7.15), in virtue of the continuity of the Δ-constant, that there exists a point x^0, $x^0 \in [x_1, x_2]$, such that $\Delta(x^0) = 0$. The last equality proves the theorem.

Taking into account a correction similar to that made in Remark 7.1, Theorem 7.2 remains valid when $m = 0$ as well. In this

case $\Delta_0(x) = \overline{f(t, x)}$, and Theorems 7.1 and 7.2 establish conditions for the existence of periodic solutions of a system

$$\frac{dx}{dt} = f(t, x)$$

by the averaged system

$$\frac{dx}{dt} = \overline{f(t, x)}$$

These conditions are weaker than those that were known before [16, 120, 160]. Thus, for systems, periodic in time of period T, of the standard form

$$\frac{dx}{dt} = \varepsilon f(t, x) \tag{7.16}$$

where ε is a small positive parameter, and $f(t, x)$ is a function continuous in t and satisfying a Lipschitz condition in x in a closed, bounded domain $D \subset E_n$, we have, from Theorem 7.1,

Theorem 7.3. *For the system of equations* (7.16) *to possess, for all sufficiently small* ε, *a periodic solution of period* T, *continuously dependent on* ε *in the neighbourhood of the point* $\varepsilon = 0$, *it is necessary that the averaged system of equations*

$$\frac{d\xi}{dt} = \varepsilon f_0(\xi) = \varepsilon \frac{1}{T} \int_0^T f(s, \xi)\, ds \tag{7.17}$$

should have the equilibrium position $\xi = \xi_0 \in D$ *and it is sufficient that this position should be an isolated one with a nonzero index.*

Proof. The sufficiency of the conditions of the theorem for the existence of a periodic solution of period T of the system of equations (7.16) ensues from the following considerations. For small ε the system of equations (7.16) is a T-system in the domain D. If a sphere with a sufficiently small radius ρ and the centre at the isolated singular point $\xi = \xi_0$ of the map $f_0(\xi)$ is taken as the domain D_1 figuring in the conditions of Theorem 7.1, then inequality (7.12) will assume the following form

$$\inf_{|\xi - \xi_0| = \rho} \varepsilon |f_0(\xi)| \geqslant \frac{\pi}{3} \mathbf{Q} |\varepsilon^2 \frac{T}{\pi} \mathbf{KM}| \tag{7.18}$$

where $\mathbf{M} = \max_{t \in [0, T], x \in D} |f(t, x)|$; and $\mathbf{K}$ is the matrix in a Lipschitz condition in $x \in D$ of the function $f(t, x)$.

Since the singular point is an isolated one, with a sufficiently small ρ

$$\inf_{|\xi - \xi_0| = \rho} |f_0(\xi)| = g > 0 \tag{7.19}$$

Consequently inequality (7.18) assumes the form

$$g \geqslant \varepsilon \frac{T}{3} |\mathbf{KM}|$$

and holds for all

$$\varepsilon \leqslant \varepsilon_0 = \frac{3g}{T|\mathbf{KM}|} \tag{7.20}$$

Therefore with $\varepsilon \leqslant \varepsilon_0$ all conditions of Theorem 7.1 hold, i. e. system (7.16) has a periodic solution $x = x(t, \varepsilon)$ of period T for which

$$|x(0, \varepsilon) - \xi_0| < \rho \tag{7.21}$$

$$|x(t, \varepsilon) - x(0, \varepsilon)| \leqslant \varepsilon \frac{T}{2} \mathbf{M} \tag{7.22}$$

The necessity of the conditions of the theorem for the existence of the periodic solution of period T continuously dependent on ε at the point $\varepsilon = 0$ of the system of equations (7.16) ensues from the following considerations. Since system (7.16) is a T-system and has a periodic solution of period T for all sufficiently small ε, the Δ-constant at the initial values $t = 0$, $x = x_0$ of these periodic solutions must be zero. It is evident that

$$\Delta(x_0) = \varepsilon f_0(x_0) + \varepsilon^2 f_1(x_0, \varepsilon)$$

where $f_1(x_0, \varepsilon)$ is a bounded function continuous for all sufficiently small ε. Therefore $\Delta(x_0) = 0$ for all sufficiently small ε only when $f_0(x_0) = 0$, which completes the proof.

8. THE FINDING OF AN APPROXIMATE PERIODIC SOLUTION

Suppose the existence of a periodic solution of a T-system has been established, say by Theorem 7.1. It is required to find it. It is evident from Corollary 7.1. that it is sufficient to calculate

the function $x_m(t, \tau, x_0)$ of the sequence (4.1) and find the point through which the periodic solution passes at $t = \tau$.

Calculating $x_m(t, \tau, x_0)$ involves integrating periodic functions. It is convenient to proceed as follows. Approximate $f(t, x)$ in some way by a function $p(t, x)$ which is a polynomial in $x = (x_1, x_2, \ldots, x_n)$. Let

$$p(t, x) = \sum_{i=1}^{l} \alpha_i(t) p_i(x) \tag{8.1}$$

where $p_i(x)$ are polynomials in x; $\alpha_i(t)$ are periodic functions, the deviation of $f(t, x)$ from $p(t, x)$ is not greater than $\mathbf{a}$, that is

$$|f(t, x) - p(t, x)| \leqslant \mathbf{a} \tag{8.2}$$

Calculate the integrals

$$a_0^i = \frac{1}{T}\int_0^T \alpha_i(t)\,dt, \qquad a_k^i = \frac{2}{T}\int_0^T k^2 \alpha_i(t) \cos k\omega t\,dt$$

$$b_k^i = \frac{2}{T}\int_0^T k^2 \alpha_i(t) \sin k\omega t\,dt \tag{8.3}$$

for $k = 1, 2, \ldots, N_i$, and $\omega = 2\pi/T$. By the calculated values $\bar{a}_k^i$ and $\bar{b}_k^i$, construct the expression

$$p_1(t, x) = \sum_{i=1}^{l} \left(\bar{a}_0^i + \sum_{k=1}^{N_i} \left(\frac{\bar{a}_k^i}{k^2} \cos k\omega t + \frac{\bar{b}_k^i}{k^2} \sin k\omega t \right) \right) p_i(x) \tag{8.4}$$

The functions $x_m(t, \tau, x_0)$ of sequence (4.1) are then found approximately from the formula

$$x_m(t, \tau, x_0) \approx x_m(t) = x_0 + \int_\tau^t \left(p_1(t, x'_{m-1}(t)) - \overline{p_1(t, x'_{m-1}(t))} \right) dt \tag{8.5}$$

This scheme is convenient because $x'_m(t)$ can be found exactly and simultaneously for all x_0, this procedure involving elementary operations of integrating trigonometric polynomials. Indeed, since $p_1(t, x)$ is a trigonometric polynomial with coefficients polynomial

in x, any of the functions $x'_m(t)$ is also a trigonometric polynomial and calculating it reduces to integrating the trigonometric polynomial.

The deviation of $x_m(t, \tau, x_0) = x_m(t)$ from $x'_m(t)$ is calculated from formula (8.5).

Let the functions $\alpha_i(t)$, $i = 1, 2, \ldots, l$, be twice piecewise differentiable, and

$$\left| \frac{d^2\alpha_i(t)}{dt^2} \right| \leqslant \mathbf{L}_i \tag{8.6}$$

We shall show that for any $m \geqslant 1$ and all $x_0 \in D'_f$ we have the inequality

$$|x_m(t, \tau, x_0) - x'_m(t)| < \alpha_1(t)\left(\sum_{i=0}^{m-1} \mathbf{Q}^i + \mathbf{B}_4\right)(\mathbf{a} + \mathbf{b}) \tag{8.7}$$

where D'_f is the set of the points of E_n contained in the domain D together with its $(T/2)(\mathbf{M} + \mathbf{a} + \mathbf{b})$-neighbourhood

$$\mathbf{b} = \max_{t,\, x \in D} |p(t, x) - p_1(t, x)|$$

$$< \sum_{i=1}^{l} \left(\delta_i\left(1 + \frac{\pi^2}{3}\right) + \frac{2N_i + 1}{\omega^2 N_i(N_i + 1)} \mathbf{L}_i\right)\mathbf{p}_i \tag{8.8}$$

$\mathbf{p}_i = \max_{x \in D} |p_i(x)|$, δ_i is the error from the calculation of integrals (8.3), and

$$\mathbf{B}_4 = \frac{1}{30\pi^3}((10\pi^3 - 30\pi^2)T\mathbf{K} + (3\pi^3 - 30\pi)T^2\mathbf{K}^2 + (\pi^3 - 30)T^3\mathbf{K}^3) \tag{8.9}$$

In fact, if $x'_m(t) \in D$ for all $m \geqslant 1$, then

$$|x_m(t) - x'_m(t)| \leqslant 2(t - \tau)\left(1 - \frac{t - \tau}{T}\right)(\mathbf{a} + \mathbf{b}) + \mathbf{K}\left(\left(1 - \frac{t - \tau}{T}\right)\int_0^t |x_{m-1}(t) - x'_{m-1}(t)|\, dt + \frac{t - \tau}{T}\int_t^{\tau + T} |x_{m-1}(t) - x'_{m-1}(t)|\, dt\right)$$

for $\tau \leqslant t \leqslant \tau + T$, from which it follows that

$$|x_m(t) - x'_m(t)|$$

$$\leqslant (\alpha_1(t) + \mathbf{K}\alpha_2(t) + \ldots + \mathbf{K}^{m-1}\alpha_m(t))(\mathbf{a} + \mathbf{b}) \tag{8.10}$$

where $\alpha_1(t), \ldots, \alpha_m(t)$ are the functions defined by relation (3.3). Taking into account Lemma 3.2 and inequality (4.12), we have from (8.10)

$$|x_m(t) - x'_m(t)| < \alpha_1(t)\left(\sum_{i=0}^{m-1} \mathbf{Q}^i + \mathbf{B}_4\right)(\mathbf{a} + \mathbf{b})$$

But $x_0 \in D'_f$; therefore, it follows from (8.5) that $|x'_1(t) - x_0| < (T/2)(\mathbf{M} + \mathbf{a} + \mathbf{b})$, i. e. $x'_1(t) \in D$.

It is easy to conclude by induction that $x'_m(t) \in D$ for all $m \geqslant 1$.

Thus inequality (8.7) is proved.

Inequality (8.8) follows from the expansion

$$p(t, x) - \overline{p(t, x)} = \sum_{i=1}^{l}\left(\sum_{k=1}^{\infty}\left(\frac{a_k^i}{k^2}\cos k\omega t + \frac{b_k^i}{k^2}\sin k\omega t\right)\right)p_i(x)$$

in virtue of the estimate $|a_k^i| + |b_k^i| \leqslant 2\mathbf{L}_i/\omega^2$.

When actually seeking the periodic solution the function $x'_m(t, \tau, x_0)$ is taken to be the calculated mth approximation to the periodic solution of the T-system $x = x(t)$ passing through the point (τ, x_0). The deviation of this approximate solution from the exact periodic solution $x(t)$ can be estimated using the inequality

$$|x(t) - x'_m(t)|$$

$$\leqslant |x(t) - x_m(t, \tau, x_0)| + |x_m(t, \tau, x_0) - x'_m(t, \tau, x_0)| \tag{8.11}$$

from which by inequalities (4.4) and (8.7), we have

$$x(t) - x'_m(t) \leqslant \alpha_1(t)\left[\mathbf{Q}^m\left((E - \mathbf{Q})^{-1} + \sum_{j=1}^{3}\delta_{jm}\mathbf{B}_j\right)\mathbf{M}\right.$$

$$\left. + \left(\sum_{i=0}^{m-1}\mathbf{Q}^i + \mathbf{B}_4\right)(\mathbf{a} + \mathbf{b})\right] \tag{8.12}$$

for all $m \geqslant 1$.

Searching for points x_0 through which the periodic solutions pass at $t = \tau$, i. e. finding points x_0 at which the Δ-constant is zero, is a time-consuming task. In some particular cases, however, the solution of this problem becomes simple. Here is one of such cases.

Theorem 8.1. *Let the right-hand side of the system*

$$\frac{dx}{dt} = f(t, x) \tag{8.13}$$

be defined in the domain

$$(t, x) \in R^1 \times D \tag{8.14}$$

and satisfy the following conditions:

(1) *the function $f(t, x)$ is periodic of period T in t, bounded and satisfies a Lipschitz condition with a matrix* $\mathbf{K}$

$$|f(t, x)| \leqslant \mathbf{M}, \qquad |f(t, x') - f(t, x'')| \leqslant \mathbf{K}|x' - x''| \tag{8.15}$$

for all $t \in R^1$ and $x, x', x'' \in D$; and

(2) *for any (t, x) from the domain* (8.14)

$$f(t, x) = -f(t, x) \tag{8.15'}$$

Then all the solutions $x = x(t)$ of system (8.13), *for which $x(0) \in D_f$, are periodic of period T. (Thus the theorem states that the domain D_f consists of the initial values of periodic solutions.)*

Proof. Given $x_0 \in D_f$, consider the successive approximations

$$x_{m+1}(t) = x_0 + \int_0^t f(t, x_m(t))\, dt, \qquad m = 0, 1, 2, \ldots \tag{8.16}$$

Since $f(t, x_0)$ is an odd function, $\overline{f(t, x_0)} = 0$. Hence

$$x_1(t) = x_0 + \int_0^t (f(t, x_0) - \overline{f(t, x_0)})\, dt = x_1(t + T)$$

i. e. the function $x_1(t)$ is periodic of period T in t. Moreover, $|x_1(t) - x_0| \leqslant (T/2)\mathbf{M}$, i. e. the function $x_1(t) \in D$. Finally, $x_1(t) = x_1(-t)$ as the integral of an odd function.

It is easy to conclude by induction that for all $m \geqslant 1$ the functions $x_m(t)$ are defined for $t \in R^1$, are periodic of period T in t, and satisfy the relation

$$x_m(t) = x_m(-t) \tag{8.17}$$

and the inequality

$$|x_m(t) - x_0| \leqslant \frac{T}{2}\mathbf{M} \tag{8.18}$$

The inequality (4.18) yields a uniform boundedness of the family $\{x_m(t)\}$, and the expression (4.16), the equicontinuity of this family. We conclude from the Arzela theorem that it is possible to choose from $\{x_m(t)\}$ a uniformly convergent subsequence

$$\{x_{m_k}(t)\}: \quad x_{m_k}(t) \underset{k\to\infty}{\to} x(t) \tag{8.19}$$

Moreover, since the conditions of the Picard theorem on the convergence of successive approximations hold, any convergent subsequences of the sequence $\{x_m(t)\}$ have one and the same limit. Consequently, the sequence $\{x_m(t)\}$ converges uniformly to the limiting function $x(t)$.

Proceeding in (8.16) to the limit convinces one that the limiting function $x(t)$ is a periodic solution of the system of equations (8.13) such that $x(0) = x_0$. Thus the theorem is proved.

In the general case the initial values of the periodic solutions of a T-system should be sought by a numerical method. It is possible to use the properties of the Δ-constant expressed by the following lemma

Lemma 8.1. *Let a T-system be given in the domain D. Suppose that D_1 is a set belonging to D_f. Then, for D_1 to have a point at which the Δ-constant is zero, it is necessary that, for some τ, all integral m's, and any $x_1 \in D_1$, the following inequality hold true:*

$$|\Delta_m(\tau, x_1)| \leqslant \sup_{x\in D_1} \mathbf{K}\left[E + \frac{T}{3}\mathbf{K}(E - \mathbf{Q})^{-1} + \mathbf{B}\right]|x - x_1| + \frac{T}{3}\mathbf{K}\mathbf{Q}^m\left[(E - \mathbf{Q})^{-1} + \sum_{j=1}^{3}\delta_{jm}\mathbf{B}_j\right]\mathbf{M} \tag{8.20}$$

in which Δ_m is the m-th approximation of the Δ-constant determined according to (7.5), $\mathbf{Q} = (T/\pi)\mathbf{K}$, *and*

$$\sup_{x\in D_1} \mathbf{K}|x - x_1| = \left\{\sup_{x\in D_1}\sum_{j=1}^{n}\mathbf{K}_{1j}|x_j - x_j'|, \ldots, \sup_{x\in D_1}\sum_{j=1}^{n}K_{nj}|x_j - x_j'|\right\}$$

Proof. Let the Δ-constant $\Delta(\tau, x)$ at the point $x \in D_1$ be zero. Then by inequality (5.11)

$$|\Delta(\tau, x_1)| < \mathbf{K}\left(E + \frac{T}{3}\mathbf{K}(E - \mathbf{Q})^{-1} + \mathbf{B}\right)|x - x_1| \qquad (8.21)$$

and in virtue of estimate (7.8)

$$|\Delta_m(\tau, x_1)|$$

$$\leqslant |\Delta(\tau, x_1)| + \frac{\pi}{3}\mathbf{Q}^{m+1}\left((E - \mathbf{Q})^{-1} + \sum_{j=1}^{3}\delta_{jm}\mathbf{B}_j\right)\mathbf{M} \qquad (8.22)$$

Combining the last two inequalities proves the validity of condition (8.20).

Taking the lemma into account, we see that to find the initial values of periodic solutions we must proceed as follows. The set D_f is decomposed into a finite number of subsets D_i. One point $x = x^{(i)}$ is chosen in each D_i, and $\Delta_m(\tau, x^{(i)})$ is calculated for an m. Comparing $\Delta_m(\tau, x^{(i)})$ and the right-hand side of inequality (8.20), we omit those of the sets D_i which satisfy the inequality

$$|\Delta_m(\tau, x^{(i)})| > \sup_{x \in D_i}\mathbf{K}\left(E + \frac{T}{3}\mathbf{K}(E - \mathbf{Q})^{-1} + \mathbf{B}\right)|x - x^{(i)}|$$

$$+ \frac{T}{3}\mathbf{K}\mathbf{Q}^m\left((E - \mathbf{Q})^{-1} + \sum_{j=1}^{3}\delta_{jm}\mathbf{B}_j\right)\mathbf{M} \qquad (8.23)$$

since these solutions do not contain any points through which periodic solutions pass at $t = \tau$.

The remaining D_i form the set $\mathfrak{M}_m^{(i)}$ through the points of which only the periodic solutions of the T-system can pass. For the case when $\Delta_m(\tau, x)$ is found for all $x \in D_f$ simultaneously, the set $\mathfrak{M}_m^{(i)}$ does not depend on i: $\mathfrak{M}_m^{(i)} = \mathfrak{M}_m$ and consists of the points satisfying the inequality

$$|\Delta_m(\tau, x)| \leqslant \frac{T}{3}\mathbf{K}\mathbf{Q}^m\left((E - \mathbf{Q})^{-1} + \sum_{j=1}^{3}\delta_{jm}\mathbf{B}_j\right)\mathbf{M} \qquad (8.24)$$

Since $\mathfrak{M}_m^{(i)}$ tends to $\mathfrak{M}$ as $i, m \to \infty$, $\mathfrak{M}$ being the initial values of the periodic solutions, any point x_1 of $\mathfrak{M}_m^{(i)}$ can be taken to be the m-th approximation to the initial value x_0 of the periodic solution.

The accuracy to which x_0 is found is then given by the inequality

$$|x_1 - x_0| \leqslant \sup_{x \in \mathfrak{M}_m^{(i)}} |x_1 - x| \tag{8.25}$$

and the deviation of the function $x_m(t, \tau, x_1)$ from the exact solution, which is natural to take as an approximate value of the periodic solution $x(t, \tau, x_0)$, by the inequality

$$\begin{aligned} &|x_m(t, \tau, x_1) - x(t, \tau, x_0)| \\ &\leqslant \sup_{x \in \mathfrak{M}_m^{(i)}} (E + \mathbf{K}(E - \mathbf{Q})^{-1} + \mathbf{KB}_4)\alpha_1(t)|x - x_1| \\ &\quad + \alpha_1(t)\mathbf{Q}^m\left((E - \mathbf{Q})^{-1} + \sum_{j=1}^{3} \delta_{jm}\mathbf{B}_j\right)\mathbf{M} \end{aligned} \tag{8.26}$$

9. FORCED PERIODIC OSCILLATIONS OF A PENDULUM SUBJECTED TO THE ACTION OF PERIODICALLY REPEATING PULSES

We shall illustrate the general points of the above method by solving the following problem by way of example.

Suppose it is necessary to investigate periodic oscillations of a simple pendulum of mass m and length l subjected to the action of periodically repeating pulses concentrated at the moments $\nu t = 2\pi, 4\pi, \ldots, 2k\pi, \ldots, k = 1, 2, \ldots$, and increasing the momentum of the pendulum at the time of a pulse by a constant value mE $(E > 0)$.

The equations of the motion of a pendulum under the action of the above forces are of the form

$$\begin{aligned} &\frac{d^2x}{dt^2} + \frac{g}{l}\sin x = 0, \qquad \nu t \neq 2k\pi \\ &\left.\frac{dx}{dt}\right|_{\nu t = 2k\pi + 0} - \left.\frac{dx}{dt}\right|_{\nu t = 2k\pi - 0} = E \end{aligned} \tag{9.1}$$

Going over from system (9.1) to the equation with a delta function [161] reduces (9.1) to the equation

$$\frac{d^2x}{dt^2} + \frac{g}{l}\sin x = \frac{E\nu}{\pi}\left(\frac{1}{2} + \sum_{k=1}^{\infty} \cos k\nu t\right) \tag{9.2}$$

and the substitution of the variables

$$x = y - \frac{E}{\pi\nu}\sum_{k=1}^{\infty}\frac{\cos k\nu t}{k^2}, \qquad \frac{dy}{dt} = z, \quad \nu t = \tau \tag{9.3}$$

to the system

$$\frac{dy}{d\tau} = \frac{z}{\nu}, \quad \frac{dz}{d\tau} = \frac{E}{2\pi} - \frac{g}{l\nu}\sin\left(y - \frac{E}{\pi\nu}\sum_{k=1}^{\infty}\frac{\cos k\tau}{k^2}\right) \tag{9.4}$$

A verification of conditions (2.5) and (2.6) shows that at

$$\nu \geqslant 2(g/l)^{1/2} \tag{9.5}$$

system (9.4) is a T-system (2π-system) in the domain

$$|y - y_1| \leqslant \alpha\pi, \qquad |z| \leqslant d_0 \tag{9.6}$$

once α and d_0 satisfy the inequalities

$$d_0 \geqslant \frac{E}{2} + \frac{g\pi}{l\nu}, \qquad \alpha \geqslant \frac{d_0}{\nu} \tag{9.7}$$

and y_1 is a fixed constant.

Suppose ν is given and α and d_0 are chosen to satisfy inequalities (9.5) and (9.7). Applying the above method of investigation to system (9.4), we can obtain the following statements:

(1) Any periodic solution of period 2π of system (9.4) passing at $\tau = \tau_0$ through the point (y_0, z_0) of the domain

$$|y - y_1| < \left(\alpha - \frac{d_0}{\nu}\right)\pi, \qquad |z| < d_0 - \left(\frac{E}{2} + \frac{g\pi}{l\nu}\right) \tag{9.8}$$

is the limit of the uniformly converging sequence of the functions

$$y_{m+1}(\tau) = y_0 + \frac{1}{\nu}\int_{\tau_0}^{\tau}(z_m(\tau) - \overline{z_m(\tau)})\,d\tau$$

$$z_{m+1}(\tau) = z_0 - \frac{g}{l\nu}\int_{\tau_0}^{\tau}\left[\left(\sin y_m(\tau) - \frac{E}{\pi\nu}\sum_{k=1}^{\infty}\frac{\cos k\tau}{k^2}\right) - \overline{\sin(\,\cdot\,)}\right]d\tau \tag{9.9}$$

(2) When

$$E > E_0 = \frac{2\pi g}{l\nu} \tag{9.10}$$

system (9.4) has no solutions, all the solutions are rotational. When

$$E \leqslant E_0/4 \tag{9.11}$$

system (9.4) has a periodic solution of period 2π, $y = y_0(\tau)$, $z = z_0(\tau)$, for all $\nu \geqslant 25$ and all $l \geqslant g$. This solution is within the domain

$$\left| y - \arcsin \frac{E}{E_0} \right| < \frac{\pi}{10}, \qquad |z| < \frac{1}{2} \tag{9.12}$$

(3) The point $y_0(0) = y_0$, $z_0(0) = z_0$ through which the periodic solution $y_0(\tau)$, $z_0(\tau)$ passes at $\tau = 0$ lies in the domain

$$\left| y - \arcsin \frac{E}{E_0} \right| < \frac{\pi}{800}, \qquad |z| < 0.01 \tag{9.13}$$

so that, with an accuracy determined by the right-hand side of inequalities (9.13), $y_0 \approx \arcsin(E/E_0)$, $z_0 \approx 0$.

(4) The solution $y_0(\tau)$, $z_0(\tau)$ is of the form

$$y_0(\tau) \approx y_0'(\tau) = \arcsin \frac{E}{E_0} - \frac{E}{\pi\nu}\left(1 - \frac{E^2}{E_0^2}\right)^{1/2}$$

$$\times \sum_{k=1}^{\infty} \frac{\cos k\tau - 1}{k^2 [l\nu^2 k^2/g - (1 - E^2/E_0^2)^{1/2}]}$$

$$z_0(\tau) \approx z_0'(\tau) = \frac{E}{\pi}\left(1 - \frac{E^2}{E_0^2}\right)^{1/2}$$

$$\times \sum_{k=1}^{\infty} \frac{\sin k\tau}{k [l\nu^2 k^2/g - (1 - E^2/E_0^2)^{1/2}]} \tag{9.14}$$

Here

$$|y_0(\tau) - y_0'(\tau)| < 0.001, \qquad |z_0(\tau) - z_0'(\tau)| < 0.05 \tag{9.15}$$

10. SCHEME OF THE METHOD FOR SECOND-ORDER EQUATIONS

Consider a second-order differential equation

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right) \tag{10.1}$$

with a right-hand side periodic of period T.

When investigating periodic solutions of this equation, it is more effective, for practical realizations, to apply the general provisions of the above numerical-analytic method directly to equation (10.1) rather than to a system of first-order equations equivalent to the original one.

With this in mind we give a scheme of the method for equations (10.1). To simplify the presentation, all calculations are made for one equation. So we consider that x of equation (10.1) is a scalar and assume the function $f(t, x, y)$ to be defined in the domain

$$R^1 \times D = R^1 \times I^1 \times I^2: \quad t \in R^1, \quad x \in [a, b] = I^1$$

$$y \in [c, d] = I^2 \tag{10.2}$$

continuous in t, x, y, periodic in t of period T, and to satisfy the inequalities

$$|f(t, x', y') - f(t, x'', y'')|$$

$$\leqslant K_1 |x' - x''| + K_2 |y' - y''|$$

$$|f(t, x, y)| \leqslant M \tag{10.3}$$

where M, K_1, K_2 are positive constants.

We define the second-order equation (10.1) as a T-system in the domain (10.2) when the constants a, b, c, d, M, K_1, K_2 are connected by the inequalities

$$b - a \geqslant \frac{T^2 M}{2}, \qquad c \leqslant -\frac{5}{6} TM \leqslant \frac{5}{6} TM \leqslant d \tag{10.4}$$

$$\frac{T}{2}\left(\frac{T}{2} K_1 + \frac{5}{3} K_2\right) < 1 \tag{10.5}$$

We introduce into consideration a linear operator L acting on the function $f(t)$, continuous for $t \in [0, T]$, by the formula

$$Lf(t) = \int_0^t (f(s) - \overline{f(t)})\, ds \tag{10.6}$$

We obtain

$$L^2 f(t) = L(Lf(t))$$

$$= \int_0^t \left(\int_0^t (f(s) - \overline{f(t)})\, ds - \overline{\int_0^t (f(s) - \overline{f(t)})\, ds} \right) dt \tag{10.7}$$

It is evident that if $f(t)$ is a periodic function of period T, then $Lf(t)$ and hence $L^2 f(t)$ are also periodic functions of period T.

According to Lemma 3.1,

$$|Lf(t)| \leqslant \alpha_1(t)\, |f(t)|_0 \tag{10.8}$$

and hence

$$|L^2 f(t)| \leqslant \alpha_1(t)\, |L(f(t))|_0$$

$$\leqslant \frac{T}{2} \alpha_1(t)\, |f(t)|_0 \leqslant \frac{T^2}{4} |f(t)|_0 \tag{10.9}$$

for all $t \in [0, T]$.

A statement similar to the main lemma for second-order equations is contained in the following lemma.

Lemma 10.1. *Let a function $f(t, x, y)$ be defined in the domain* (10.2), *continuous in t, x, y, periodic in t of period T and satisfy inequalities* (10.3) *to* (10.5). *Then the sequence of functions periodic in t of period T,*

$$x_{m+1}(t, x_0) = x_0 + L^2 f\left(t, x_m(t, x_0), \frac{dx_m(t, x_0)}{dt}\right) \tag{10.10}$$

$$m = 0, 1, 2, \ldots$$

converges when $m \to \infty$ uniformly in

$$t, x_0 \in R^1 \times I_f^1 = R^1 \times \left[a + \frac{T^2 M}{4}, b - \frac{T^2 M}{4} \right] \tag{10.11}$$

to the function $x^0(t, x_0)$ *defined in the domain* (10.11), *periodic in* t *of period* T *and satisfying the equation*

$$x(t) = x_0 + L^2 f\left(t, x(t), \frac{dx(t)}{dt}\right) \tag{10.12}$$

Here

$$\begin{pmatrix} |x^0(t, x_0) - x_m(t, x_0)| \\ \left|\frac{dx^0(t, x_0)}{dt} - \frac{dx_m(t, x_0)}{dt}\right| \end{pmatrix} \leqslant \mathbf{Q}_0^m (E - \mathbf{Q}_0)^{-1} \mathbf{z}_1 \tag{10.13}$$

where

$$\mathbf{Q}_0 = \frac{T}{2}\begin{pmatrix} \frac{T}{2}K_1 & \frac{T}{2}K_2 \\ \frac{5}{3}K_1 & \frac{5}{3}K_2 \end{pmatrix} \tag{10.14}$$

$$\mathbf{z}_1^0 \leqslant \frac{T}{2} M \begin{pmatrix} \frac{T}{2} \\ \frac{5}{3} \end{pmatrix} \tag{10.15}$$

Proof. We show by induction that $x_m(t, x_0)$ and $dx_m(t, x_0)/dt$, when $t \in R^1$, take values in the domain (10.2) for all $m = 0, 1, 2, \ldots$. Indeed, by inequality (10.9), when $m = 0$, expression (10.10) yields

$$|x_1(t, x_0) - x_0| \leqslant |L^2 f(t, x_0, 0)| \leqslant \frac{T^2 M}{4}$$

so that $a \leqslant x_1(t, x_0) \leqslant b$ for all $t \in R^1$ once $x_0 \in I_f^1$. Moreover, on differentiating $x_1(t, x_0)$ we find

$$\frac{dx_1(t, x_0)}{dt} = Lf(t, x_0, 0) - \overline{Lf(t, x_0, 0)}$$

hence

$$\left|\frac{dx_1(t, x_0)}{dt}\right| \leqslant \alpha_1(t) M + \frac{1}{T}\int_0^T \alpha_1(t) M \, dt$$

$$\leqslant \left(\alpha_1(t) + \frac{T}{3}\right) M \leqslant \frac{5TM}{6}$$

By condition (10.4), it follows from the last inequality that $c \leqslant dx_1(t, x_0)/dt \leqslant d$ when $t \in R^1$.

Suppose that $a \leqslant x_m(t, x_0) \leqslant b$ and $c \leqslant dx_m(t, x_0)/dt \leqslant d$ for all m's less than an integer k, $k \geqslant 1$, and all $t \in R^1$. Taking into account the recurrence relation defining $x_{k+1}(t, x_0)$, we find that

$$|x_{k+1}(t, x_0) - x_0| \leqslant \left| L^2 f\left(t, x_k(t, x_0), \frac{dx_k(t, x_0)}{dt}\right)\right|_0$$

$$\leqslant \alpha_1(t) \frac{TM}{2} \leqslant \frac{T^2 M}{4}$$

$$\left|\frac{dx_{k+1}(t, x_0)}{dt}\right| \leqslant \left(\alpha_1(t) + \frac{T}{3}\right)\left|f\left(t, x_k(t, x_0), \frac{dx_k(t, x_0)}{dt}\right)\right|_0$$

$$\leqslant \frac{5\,TM}{6}$$

so that $a \leqslant x_{k+1}(t, x_0) \leqslant b$ and $c \leqslant dx_{k+1}(t, x_0)/dt \leqslant d$ for all $t \in R^1$ once $x_0 \in I_f^1$.

It follows from the above that the functions of sequence (10.10) satisfy the inequalities

$$a \leqslant x_m(t, x_0) \leqslant b, \qquad c \leqslant \frac{dx_m(t, x_0)}{dt} \leqslant d \tag{10.16}$$

for all $m = 0, 1, 2, \ldots, t \in R^1$, $x_0 \in I_f^1$.

To prove the convergence of sequence (10.10), we estimate the difference $|x_{m+1}(t, x_0) - x_m(t, x_0)|$. Denote by $w_m(t)$ the expression

$$w_m(t) = f\left(t, x_m(t, x_0), \frac{dx_m(t, x_0)}{dt}\right) - f\left(t, x_{m-1}(t, x_0), \frac{dx_{m-1}(t, x_0)}{dt}\right)$$

We have

$$|x_{m+1}(t, x_0) - x_m(t, x_0)|$$

$$\leqslant |L^2 w_m(t)| \leqslant \alpha_1(t)|Lw_m(t)|_0 \leqslant \alpha_1(t)\frac{T}{2}|w_m(t)|_0$$

$$\leqslant \alpha_1(t)\frac{T}{2}\left(K_1|x_m(t, x_0) - x_{m-1}(t, x_0)|_0\right.$$

$$\left. + K_2\left|\frac{dx_m(t, x_0)}{dt} - \frac{dx_{m-1}(t, x_0)}{dt}\right|_0\right) \tag{10.17}$$

On differentiating identity (10.10) we also find

$$\left|\frac{dx_{m+1}(t, x_0)}{dt} - \frac{dx_m(t, x_0)}{dt}\right|$$

$$\leqslant |Lw_m(t) - \overline{Lw_m(t)}| \leqslant \left(\alpha_1(t) + \frac{T}{3}\right)|w_m(t)|_0$$

$$\leqslant \left(\alpha_1(t) + \frac{T}{3}\right)\left(K_1 |x_m(t, x_0) - x_{m-1}(t, x)|_0 + K_2 \left|\frac{dx_m(t, x_0)}{dt} - \frac{dx_{m-1}(t, x_0)}{dt}\right|_0\right) \tag{10.18}$$

Rewrite inequalities (10.17) and (10.18) in vector form

$$\mathbf{z}_{m+1}(t) \leqslant Q(t)\mathbf{z}_m^0 \tag{10.19}$$

where

$$\mathbf{z}_{m+1}(t) = \begin{pmatrix} |x_{m+1}(t, x_0) - x_m(t, x_0)| \\ \left|\frac{dx_{m+1}(t, x_0)}{dt} - \frac{dx_m(t, x_0)}{dt}\right| \end{pmatrix}$$

$$Q(t) = \begin{pmatrix} \alpha_1(t)\frac{T}{2}K_1 & \alpha_1(t)\frac{T}{2}K_2 \\ \left(\alpha_1(t) + \frac{T}{3}\right)K_1 & \left(\alpha_1(t) + \frac{T}{3}\right)K_2 \end{pmatrix}$$

$$\mathbf{z}_m^0 = \begin{pmatrix} |x_m(t, x_0) - x_{m-1}(t, x_0)|_0 \\ \left|\frac{dx_m(t, x_0)}{dt} - \frac{dx_{m-1}(t, x_0)}{dt}\right|_0 \end{pmatrix}, \qquad \mathbf{z}_1^0 \leqslant \frac{T}{2}M\begin{pmatrix} \frac{T}{2} \\ \frac{5}{3} \end{pmatrix}$$

It follows from inequality (10.19) that

$$\mathbf{z}_{m+1}^0 \leqslant \mathbf{Q}_0 \mathbf{z}_m^0 \tag{10.20}$$

where

$$\mathbf{Q}_0 = \frac{T}{2}\begin{pmatrix} \frac{T}{2}K_1 & \frac{T}{2}K_2 \\ \\ \frac{5}{3}K_1 & \frac{5}{3}K_2 \end{pmatrix} \tag{10.21}$$

By iterating inequality (10.20) we have

$$\mathbf{z}_{m+1}^0 \leqslant \mathbf{Q}_0^m \mathbf{z}_1^0 \tag{10.22}$$

which leads to the estimate

$$\sum_{i=1}^{m} \mathbf{z}_1^0 \leqslant \sum_{i=1}^{m} \mathbf{Q}_0^{i-1} \mathbf{z}_1^0 \tag{10.23}$$

Since the matrix $\mathbf{Q}_0$ has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = (T/2) \times \times (TK_1/2 + 5K_2/3) < 1$, the series (10.23) is uniformly convergent:

$$\lim_{m\to\infty} \sum_{i=1}^{m} \mathbf{Q}_0^{i-1} \mathbf{z}_1^0 = \sum_{i=1}^{\infty} \mathbf{Q}_0^{i-1} \mathbf{z}_1^0 = (E - \mathbf{Q}_0)^{-1} \mathbf{z}_1^0 \tag{10.24}$$

The limiting relation (10.24) signifies a uniform convergence of the sequence $[x_m(t, x_0), dx_m(t, x_0)/dt]$

$$\lim_{m\to\infty} x_m(t, x_0) = x^0(t, x_0)$$

$$\lim_{m\to\infty} \frac{dx_m(t, x_0)}{dt} = \frac{dx^0(t, x_0)}{dt} \tag{10.25}$$

By inequality (10.22), the estimate

$$\begin{pmatrix} |x^0(t, x_0) - x_m(t, x_0)| \\ \\ \left| \frac{dx^0(t, x_0)}{dt} - \frac{dx_m(t, x_0)}{dt} \right| \end{pmatrix} \leqslant \mathbf{Q}_0^m (E - \mathbf{Q}_0)^{-1} \mathbf{z}_1^0 \tag{10.26}$$

is obtained for the deviation of $(x_m(t, x_0), dx_m(t, x_0)/dt)$ from $(x^0(t, x_0), dx^0(t, x_0)/dt)$.

Proceeding in the recurrence relation (10.10) to the limit as $m \to \infty$ convinces us that the limiting function $x^0(t, x_0)$ satisfies equation (10.12), which completes the proof of the lemma.

As before, the parameter μ for which the solution of the equation

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right) - \mu \tag{10.27}$$

which takes at $t = 0$ the value of $x = x_0$, is periodic of period T provided the value of μ is unique, will be called the Δ-constant at the point 0, x_0 with respect to equation (10.1).

Lemma 10.1 readily yields the statement of the existence and properties of the Δ-constant for the points $x_0 \in I_f^1$ of equation (10.1) in the case where it is a T-system in the domain D.

This makes it possible to consider the periodic problem of control for equation (10.1), as well as for the system of first-order equations: to choose a parameter μ and an initial velocity $dx(0)/dt = y_0$ for equation (10.27) so that a solution of this equation passing at $t = 0$ through the point $x = x_0$ is periodic of period T. The solution of this problem, on the assumption that equation (10.1) is a T-system and that $x_0 \in I_f^1$, is unique and given by the control μ equal to the value of the Δ-constant at the point $(0, x_0)$,

$$\mu = \Delta(x_0) = \overline{f\left(t, x^0(t, x_0), \frac{dx^0(t, x_0)}{dt}\right)} \tag{10.28}$$

with the initial velocity

$$y_0 = -\int_0^t \overline{\left(f\left(s, x^0(s, x_0), \frac{dx^0(s, x_0)}{ds}\right) - \Delta(x_0)\right)} ds \tag{10.29}$$

where $x^0(t, x_0)$ is the limit of sequence (10.10).

An approximate calculation of both the control μ and the initial velocity y_0 can be carried out according to formulas (10.28) and (10.29) where $x_m(t, x_0)$ is substituted for $x^0(t, x_0)$.

In addition, Lemma 10.1 describes an algorithm for constructing an approximate periodic solution of equation (10.1) which is a

T-system, provided a point $x = x_0$ is known through which this solution passes at $t = 0$.

Theorem 10.1. *Let equation* (10.1) *be a T-system in the domain D. Then, if equation* (10.1) *has a periodic solution of period T, i. e.* $x = x(t)$, *which takes at* $t = 0$, *the value* x_0, $x_0 \in I_f^1 = [a + T^2M/4,\ b - T^2M/4]$, *we have*

$$x(t) = x^0(t,\ x_0) \tag{10.30}$$

where $x^0(t,\ x_0)$ *is the limit of the sequence of functions* $x_m(t,\ x_0)$ *defined according to* (10.10).

It follows immediately from the statement of the equality theorem (10.30) *that the initial value of the velocity* $dx(0)/dt = y_0$, *which together with* x_0 *determines the periodic solution, satisfies the inequality* $|y_0| \leqslant 5\,TM/6$.

Proof. Indeed, if $x(t)$ is a periodic solution of period T of equation (10.1), then

$$\overline{f\left(t, x(t), \frac{dx(t)}{dt}\right) = 0}, \qquad \frac{dx(0)}{dt} = \overline{-\int_0^t f\left(s, x(s), \frac{dx(s)}{ds}\right) ds} \tag{10.31}$$

hence

$$x(t) = x_0 + \frac{dx(0)}{dt} t + \int_0^t \int_0^t f\left(s, x(s), \frac{dx(s)}{ds}\right) ds\, dt$$

$$= x_0 + L^2 f\left(t, x(t), \frac{dx(t)}{dt}\right)$$

The last equality proves that $x(t)$ is a solution of equation (10.12).

The conditions of Theorem 10.1 ensure the uniqueness of the solution of equation (10.12) for the points $x_0 \in I_f^1$. Now we show it.

Suppose the contrary. There would then be at least two different solutions, $x_t = x(t,\ x_0)$ and $y_t = y(t,\ x_0)$, of equation (10.12). For their difference we should obtain the identity

$$|x_t - y_t| = \left| L^2 \left(f\left(t, x_t, \frac{dx_t}{dt}\right) - f\left(t, y_t, \frac{dy_t}{dt}\right)\right)\right| \tag{10.32}$$

by estimating it we should find

$$|x_t - y_t| \leqslant \alpha_1(t)\frac{T}{2}\left(K_1|x_t - y_t|_0 + K_2\left|\frac{dx_t}{dt}\ \frac{dy_t}{dt}\right|_0\right) \tag{10.33}$$

On differentiating (10.32) we should also obtain

$$\left|\frac{dx_t}{dt} - \frac{dy_t}{dt}\right| \leqslant \left(\alpha_1(t) + \frac{T}{3}\right)\left(K_1|x_t - y_t|_0 + K_2\left|\frac{dx_t}{dt} - \frac{dy_t}{dt}\right|_0\right) \tag{10.34}$$

Inequalities (10.33) and (10.34) would lead to the estimate

$$\begin{pmatrix} |x_t - y_t|_0 \\ \left|\frac{dx_t}{dt} - \frac{dy_t}{dt}\right|_0 \end{pmatrix} \leqslant \mathbf{Q}_0 \begin{pmatrix} |x_t - y_t|_0 \\ \left|\frac{dx_t}{dt} - \frac{dy_t}{dt}\right|_0 \end{pmatrix}$$

by iterating which we should find that

$$\begin{pmatrix} |x_t - y_t|_0 \\ \left|\frac{dx_t}{dt} - \frac{dy_t}{dt}\right|_0 \end{pmatrix} \leqslant \mathbf{Q}_0^m \begin{pmatrix} |x_t - y_t|_0 \\ \left|\frac{dx_t}{dt} - \frac{dy_t}{dt}\right|_0 \end{pmatrix}$$

But $\mathbf{Q}_0^m \to 0$ as $m \to \infty$, so that proceeding in the last inequality to the limit we should obtain the equalities $x_t = y_t$, $dx_t/dt = dy_t/dt$ which contradict the supposition.

It follows immediately from the uniqueness of the solution of equation (10.12) that $x(t) = x^0(t, x_0)$. Thus the theorem is proved.

It is evident from equation (10.12) that any of its solutions for which

$$\overline{\Delta(x_0) = f\left(t, x^0(t, x_0), \frac{dx^0(t, x_0)}{dt}\right) = 0} \tag{10.35}$$

is a periodic solution of equation (10.1) such that

$$x(0) = x_0, \quad \frac{dx(0)}{dt} = -\int_0^t f\left(s, x^0(s, x_0), \frac{dx^0(s, x_0)}{ds}\right) ds \tag{10.36}$$

Therefore, the problem of the existence of a periodic solution of period T of equation (10.1) is uniquely connected with that of the existence of zeros of the function $\Delta(x_0)$ of the form (10.35).

Since the function $\Delta(x_0)$ is found only approximately, from the sequence of functions

$$\Delta_m(x_0) = f\left(t, x_m(t, x_0), \frac{dx_m(t, x_0)}{dt}\right) \tag{10.37}$$

the former problem arises as to how to draw conclusions concerning the zeros of the function $\Delta(x_0)$ from those of the function $\Delta_m(x_0)$. To solve this problem, one should take into account that each of the functions $\Delta_m(x_0)$, as well as the limiting function $\Delta(x_0)$, is defined and continuous for x_0 in the interval I_f^1 and satisfies the inequality

$$|\Delta_m(x_0) - \Delta(x_0)|$$

$$\leqslant \frac{1}{T}\int_0^T \left| f\left(s, x_m(s,x_0), \frac{dx_m(s,x_0)}{ds}\right) - f\left(s, x^0(s, x_0), \frac{dx^0(s,x_0)}{ds}\right)\right| ds$$

$$\leqslant K_1 |x^0(t,x_0) - x_m(t,x_0)|_0 + K_2 \left| \frac{dx^0(t,x_0)}{dt} - \frac{dx_m(t,x_0)}{dt}\right|$$

$$\leqslant \left\langle \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}, \mathbf{Q}_0^m (E - \mathbf{Q}_0)^{-1} \mathbf{z}_1^0 \right\rangle = d_m \tag{10.38}$$

where $\langle\cdot\rangle$ denotes the ordinary scalar product in the space E_2. This results in the statement similar to that of Theorem 7.2:

Theorem 10.2. *Let equation* (10.1) *be a T-system in the domain* (10.2). *Suppose that for an* $m \geqslant 0$ *the function* $\Delta_m(x_0)$ *defined according to formula* (10.37) *satisfies the inequalities*

$$\min_{x_0 \in I_f^1} \Delta_m(x_0) \leqslant -d_m, \qquad \max_{x_0 \in I_f^1} \Delta_m(x_0) \geqslant d_m \tag{10.39}$$

where $I_f^1 = [a + T^2M/4, b - T^2M/4]$; *and* d_m *is the constant* (10.38). *Then equation* (10.1) *has a periodic solution of period* T, $x = x(t)$, *for which* $x(t)$, $dx(t)/dt \in D$ *when* $t \in R^1$

$$x(0) \in I_f^1, \qquad \left|\frac{dx(0)}{dt}\right| \leqslant \frac{5TM}{6} \tag{10.40}$$

The fact that the function $\Delta(x_0)$ is a scalar function of one argument allows one to exploit the classical theorem on the zeros of an analytic function [180] and to obtain the following statement.

Theorem 10.3. *If the right-hand side of equation* (10.1) *is a polynomial in* x *and* $y = dx/dt$ *and satisfies inequalities* (10.3) *to* (10.5), *then, if equation* (10.1) *has a periodic solution of period* T, *it has one periodic solution either for each value* $x(0) = x_0$ *in the interval* I_f^1, *or only for a finite number of such values.*

Indeed, since by formula (10.29) each zero of the function $\Delta(x_0)$ has the corresponding unique periodic solution of period T of equation (10.1), the number of periodic solutions of the equation is determined by that of the zeros of the function $\Delta(x_0)$. Under the assumptions of Theorem 10.3, the function $\Delta(x_0)$ is analytic, being the limit in a strip of the complex domain of a sequence of polynomials, uniformly convergent and containing the interval I_f^1. It is, therefore, either an identical zero in the interval I_f^1 or has a finite number of zeros in it.

In the former case, the initial values $x(0) = x_0$, $y_0 = dx(0)/dt$ of the periodic solutions fill some curve of the rectangle

$$D_f = I_f^1 \times I_f^2, \qquad I_f^2 = \left[c + \frac{5TM}{6},\ d - \frac{5\,TM}{6}\right] \tag{10.41}$$

and in the latter they are its isolated points.

As already established, finding the initial values of periodic solutions of equation (10.1) which is a T-system in the domain (10.2) is equivalent to seeking the zeros of the function $\Delta(x_0)$.

According to (10.38) we have

$$|\Delta_m(x^0)| \leqslant d_m, \qquad m = 0,\ 1,\ 2, \ldots \tag{10.42}$$

for each zero x^0 of the function $\Delta(x_0)$.

All zeros of the function $\Delta(x_0)$ are, consequently, contained in the solutions of inequality (10.42). If one denotes by $\mathfrak{D}_m$ a set of points in the interval I_f^1 satisfying inequality (10.42), then any point of the set may be zero of the function $\Delta(x_0)$. But $\mathfrak{D}_m$ with

$m \to \infty$ tends to the set $\mathfrak{D}$ of the zeros of the function $\Delta(x_0)$; therefore, any point of $\mathfrak{D}_m$ can be taken to be the m-th approximation to the initial value $x(0) = x_0$ of a periodic solution.

Thus the problem of finding the initial values $x(0) = x_0$ of the periodic solutions of equation (10.1) reduces to finding the zeros of the function $\Delta_m(x_0)$ or, if there are none, to searching for points satisfying inequality (10.42).

In a number of cases exact initial values of some periodic solutions of equation (10.1) can be found. This is possible, for example, when the right-hand side of equation (10.1) is such that

$$\Delta_m(x) = 0 \tag{10.43}$$

for all $m = 0, 1, \ldots$ and for a certain fixed $x = x^0$.

Just as in the case of systems of first-order equations, equality (10.43) always holds for those equations whose right-hand sides are odd in t. This leads to the following theorem.

Theorem 10.4. *Let equation* (10.1) *be a T-system in the domain* (10.2). *Suppose that for all* t, x, y *of this domain*

$$f(-t, x, y) = -f(t, x, y) \tag{10.44}$$

Then equation (10.1) *has the periodic solution* $x(t)$ *of period* T *which is determined by the relation* $x(t, 0) = x^0(t, 0)$.

Proof. According to the statements of Lemma 10.1, the sequence $x_m(t, 0)$ is uniformly convergent. Since the function $f(t, x, y)$ is odd in t, this sequence is odd itself

$$x_m(-t, 0) = -x_m(t, 0), \qquad m = 1, 2, \ldots$$

and hence

$$f\left(-t, x_m(-t, 0), \frac{dx_m(-t, 0)}{dt}\right)$$

$$= f\left(-t, -x_m(t, 0), \frac{dx_m(t, 0)}{dt}\right) = -f\left(t, x_m(t, 0), \frac{dx_m(t, 0)}{dt}\right)$$

$$m = 0, 1, 2, \ldots$$

But then $\Delta_m(0)$, being a mean of an odd function, is zero; consequently, $\Delta(0) = 0$, which completes the proof.

Note that Theorem 10.4 supplements some results [53, 139] concerning the investigation of periodic solutions of systems possessing certain symmetry.

For one example, consider torsional harmonic oscillations of a shaft consisting of two nonlinearly elastically connected masses one of which is acted upon by a sinusoidal torsional moment $F(t) = E\cos \nu t$. The problem of oscillations of such a shaft leads to the equation of the form [16]

$$I_1 I_2 \frac{d^2\theta}{dt^2} + (I_1 + I_2)c(\theta) = f\left(\frac{d\theta}{dt}\right) = E\cos \nu t \qquad (10.45)$$

where I_1 and I_2 are the moments of the inertia of the masses, $\theta = \theta_1 - \theta_2$ is the angle of torsion, and f is the function of velocity allowing for friction.

If one considers the moments of inertia to be great, equation (10.45) belongs to the form under consideration, for inequalities (10.4) and (10.5) can always be satisfied for fixed a, b, c, d. Therefore, any periodic solution of period 2π in νt of equation (10.45) can be found as the limit of the sequence of functions (10.10). Thus in a first approximation the periodic solutions of equation (10.45) have the form

$$\theta(t) = \theta_0 + \frac{E}{I_1 I_2 \nu^2}(1 - \cos \nu t) \qquad (10.46)$$

if $\theta(0) = \theta_0$ is the initial value of the periodic solution. It can be seen from (10.47) that the extreme values of the torsion angle of a shaft decrease as the frequency of the torsional moment increases and remain unchanged if E and ν^2 change keeping the ratio E/ν^2 constant.

As

$$\Delta_0(\theta) = -\frac{I_1 + I_2}{I_1 I_2} c(\theta) - \frac{f(0)}{I_1 I_2} \qquad (10.47)$$

and is, consequently, of the first order of smallness with respect to I_1^{-1}, I_2^{-1} provided $c(\theta) \neq 0$ and d_0 is of the second order of smallness, inequalities (10.39) of Theorem 10.2 always hold once the function $(I_1 + I_2)c(\theta) + f(0)$ takes values with different signs. But (10.47) can take values with different signs at large I_1, I_2 provided the different signs are taken by the function $c(\theta)$. Hence the conclusion: equation (10.45) has always periodic solutions for large I_1, I_2 as soon as $c(\theta)$ takes values with different signs. The initial values of the periodic solutions are, in a first approximation, the solutions

of the equation

$$(I_1 + I_2)c(\theta) + f(0) = 0 \tag{10.48}$$

Taking into account Theorem 10.3 and inequality (10.42) we also conclude that equation (10.45) has a finite number of periodic solutions if $c(\theta)$ and $f(d\theta/dt)$ are polynomials with respect to θ, $d\theta/dt$ and $c(\theta) \not\equiv 0$.

It should be noted that going over from one equation (10.1) to systems of n equations of such type introduces some changes to the statements of the present section.

First of all, the conditions separating a T-system in the domain

$$x \in D_1 \subset E_n, \qquad y \in D_2 = I_1 \times I_2 \times \ldots \times I_n$$

$$I_j = [c_j, d_j], \qquad j = 1, 2, \ldots, n \tag{10.49}$$

from (10.1) are defined by the vector $\mathbf{M}$ and the matrices $\mathbf{K}_1$, $\mathbf{K}_2$ of the inequality

$$|f(t, x, y)| \leqslant \mathbf{M}, \qquad t, x, y \in R^1 \times D_1 \times D_2$$

$$|f(t, x', y') - f(t, x'', y'')| \leqslant \mathbf{K}_1 |x' - x''| + \mathbf{K}_2 |y' - y''|$$

$$x', x'' \in D_1, \qquad y', y'' \in D_2 \tag{10.50}$$

and consist in the following:

(1) the set D_f^1 consisting of the points contained in the domain D_1 together with their $(T^2/4)\mathbf{M}$-neighbourhoods is nonempty:

$$D_f^1 \neq \varnothing \tag{10.51}$$

and

$$c_j \leqslant -\frac{5}{6}TM_j \leqslant \frac{5}{6}TM_j \leqslant d_j, \qquad j = 1, 2, \ldots, n \tag{10.52}$$

(2) the largest eigenvalue of the matrix

$$\mathbf{Q}_0 = \frac{T}{2}\begin{pmatrix} \frac{T}{2}\mathbf{K}_1 & \frac{T}{2}\mathbf{K}_2 \\ \frac{5}{3}\mathbf{K}_1 & \frac{5}{3}\mathbf{K}_2 \end{pmatrix} \tag{10.53}$$

is less than unity:

$$\lambda_{\mathbf{Q}_0} \leqslant 1 \tag{10.54}$$

Theorem 10.2 on periodic solutions is formulated in the same way as Theorem 7.1, the only difference being that the right-hand side of inequality (7.7) should be replaced by the constant $|d_m|$, where

$$\mathbf{d}_m = (\mathbf{K}_1\mathbf{K}_2)\,\mathbf{Q}_0^m(E - \mathbf{Q}_0^m)^{-1}\mathbf{Z}_1^0 \tag{10.55}$$

$$\mathbf{Z}_1^0 = \begin{pmatrix} \dfrac{T}{2} & \dfrac{T}{2}\mathbf{M} \\ & \dfrac{5}{3}\mathbf{M} \end{pmatrix} \tag{10.56}$$

Finally, Theorem 10.3 does not hold for the multidimensional case.

11. ROTARY MOTIONS OF AUTONOMOUS SYSTEMS

Consider a system whose mathematical model is a second-order differential equation

$$\frac{d^2x}{dt^2} = f\left(x, \frac{dx}{dt}\right), \qquad x \in E_1 \tag{11.1}$$

with the right-hand side given by a function $f(x, y)$ periodic in x of period 2π.

If one interprets x as an angular coordinate, i. e. if one considers the points x and $x + 2k\pi$, $k = 1, 2, \ldots$ as identical, then it is natural to call a motion $x = x(t)$ of system (11.1) a rotary one if the function $x(t)$ satisfies the condition of periodicity of the second kind

$$x(t + T) = \nu T + x(t), \qquad t \in R^1 \tag{11.2}$$

where T and ν are positive constants, the value $2\pi\nu$ being called the rotation number and T the rotation period [140].

Equality (11.2) leads to the expression

$$x(t) = \nu t + \Phi(\nu t) \tag{11.3}$$

where $\Phi(\varphi)$ is a function periodic in φ of period νT.

Indeed, since

$$t = \left\{\left(\frac{t}{T}\right) + \left[\frac{t}{T}\right]\right\} T$$

where $\left(\frac{t}{T}\right)$ is the fractional and $\left[\frac{t}{T}\right]$ the integral part of the number t/T, we find that according to (11.2)

$$x(t) = x\left(\left(\frac{t}{T}\right)T + \left[\frac{t}{T}\right]T\right) = \nu\left[\frac{t}{T}\right]T + x\left(\left(\frac{t}{T}\right)T\right)$$

$$= \nu t + x\left(\left(\frac{t}{T}\right)T\right) - \nu\left(\frac{t}{T}\right)T = \nu t + \Phi(\nu T)$$

with $\Phi(\nu t) = x\left(\left(\frac{t}{T}\right)T\right) - \nu\left(\frac{t}{T}\right)$ being a function periodic in t of period T.

We shall elucidate the conditions under which the numerical-analytic method can be applied to finding rotary motions. We shall first establish some general properties of the rotary motions of system (11.1). Suppose the function $f(x, y)$ is continuous in x, y and satisfies a Lipschitz condition in the domain

$$x \in R^1, \qquad a \leqslant y \leqslant b \tag{11.4}$$

so that there are constants M, K, and K_1 such that for all x, x', x'', y, y', y'' in the domain (11.4)

$$|f(x, y)| \leqslant M \tag{11.5}$$

$$|f(x', y') - f(x'', y'')| \leqslant K_1 |x' - x''| + K_2 |y' - y''| \tag{11.6}$$

Besides the solution $x = x(t)$ of the equation we shall consider in the phase space (x, y) the trajectories of the system which are the integral curves of the equation

$$y\,dx = f(x, y)\,dx \tag{11.7}$$

The following statements hold under the assumptions made by us about the right-hand side of equation (11.1):

Theorem 11.1. *A phase trajectory* $y = y(x)$ *of a rotary motion* $x = x(t)$ *of system* (11.1) *does not intersect the* x *axis.*

Proof. Let $y = y(x)$ be a phase trajectory of a rotary motion. By equality (11.3) it is defined for all $x \in R^1$. Let it be, at $t = \tau_0$, at a point A outside the x axis (Figure 1). Suppose at some $t = \tau_1 > \tau_0$ the phase trajectory intersects the x axis at a point $B(x_1, 0)$. First of all we shall show that $f(x_1, 0) \neq 0$.

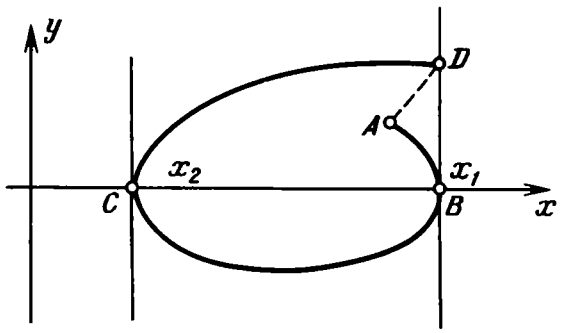

Fig. 1

Indeed, if $f(x_1, 0) = 0$, then $x = x_1$ would be an equilibrium state of equation (11.1) and the phase trajectory $y = y(x)$, on reaching the point B at $t = \tau_1$ would remain there for all $t > \tau_1$, which is impossible, for then $x = x(t)$ would not run through all the values in R^1.

In the neighbourhood of the point $B(x_1, 0)$ the phase trajectory coincides with the integral curve of the equation

$$\frac{dx}{dy} = \frac{y}{f(x, y)} \tag{11.8}$$

Since, by equation (11.8), $dx(y)/dy = 0$ with $y = 0$ and $dx(y)/dy$ changes the sign when passing through the point $y = 0$ (in the neighbourhood of the point B the function $f(x, y)$ has the sign of $f(x_1, 0)$), the integral curve $x(y)$ and hence the phase trajectory $y = y(x)$ intersecting the axis x remains on one side of the straight line $x = x_1$. To get to the other side of the straight line $x = x_1$ (and this is necessary for the phase trajectory $y(x)$, for it is defined for all $x \in R^1$), the phase trajectory must first intersect the axis x and then the straight line $x = x_1$, but already with $y \neq 0$. As t continues to increase, the phase trajectory may remain on the same side of the straight line $x = x_1$ for all t, or it may intersect the x axis at some point $C(x(\tau_2), 0)$. In the first case when t decreases from τ_0 the phase trajectory cannot get to the other side of the straight line $x = x_1$ in virtue of the fact that it cannot cross itself nor can it have vertical tangents in the half-planes $y > 0$ or $y < 0$. This contradicts the fact that $y(x)$ is defined for all $x \in R^1$. If, however, as t increases the phase trajectory intersects the x axis at some point $C(x_2, 0)$, then in virtue of the fact that it must remain on one side of the straight line $x = x_2$, it may get to the

other side of the straight line $x = x_1$ only by getting to a point $D(x_1, y \neq 0)$ of the straight line $x = x_1$. This is also impossible because as t decreases the phase trajectory cannot leave the domain $ABCDA$, which also contradicts the tendency of x to $-\infty$ at $t \to \infty$. This proves the theorem.

Theorem 11.2. *A motion $x = x(t)$ of system (11.1) is rotary if and only if its phase trajectory $y = y(x)$ is periodic of period $2\pi/k$, where k is an integer.*

Proof. Let $x(t)$, $x(0) = x_0$ be a rotary motion, then according to equation (11.3) the function

$$y(t) = \frac{dx}{dt} = \nu + \frac{d\Phi(\nu t)}{dt}$$

is periodic, $y(t + T) = y(t)$, of period T. For the phase trajectory we have

$$y(x(t), x_0) = y(t) = y(t + T) = y(x(t + T), x_0)$$

$$= y(\nu T + x(t), x_0) \tag{11.9}$$

and since $x(t)$ at $t > 0$ runs over all the values of the interval (x_0, ∞), it follows from relation (11.9) that $y(x, x_0) = y(x + \nu T, x_0)$, i. e. the function $y(x, x_0)$ is periodic of period νT.

Moreover, as according to Theorem 11.1 $y(x) \neq 0$, equation (11.7) is a first-order equation without singularities and by Massera's theorem [110] its periodic solution $y = y(x)$ is of period $T_1 = \nu T = 2\pi/k$, where k is a positive integer.

Now let the phase trajectory $y = y(x)$ for the solution $x = x(t)$ be periodic of period 2π. Note first of all that $y(x) \neq 0$. Indeed, if we had $y(x) = 0$ at some point, then, using the same reasoning as in the proof of Theorem 11.1, we should come to a contradiction with the periodicity of the function $y(x)$.

Since $y = dx/dt$, the motion of system (11.1) corresponding to the phase trajectory $y(x)$ is defined as the periodic solution of the equation

$$\frac{dx}{dt} = y(x) \tag{11.10}$$

The function $y(x)$ is periodic in x and continuously differentiable; therefore, there exists a solution of equation (11.10) for all $t \in R^1$.

On integrating equation (11.10) we have

$$t = \int_{x_0}^{x} \frac{dx}{y(x)} = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{dx}{y(x)} (x - x_0) + \psi(x) \tag{11.11}$$

where

$$\psi(x) = \int_{x_0}^{x} \frac{dx}{y(x)} - \frac{1}{2\pi} \int_{0}^{2\pi} \frac{dx}{y(x)} (x - x_0) \tag{11.12}$$

Let

$$\nu = \left(\frac{1}{2\pi} \int_{0}^{2\pi} \frac{dx}{y(x)} \right)^{-1} \tag{11.13}$$

It then follows from (11.11) that with $\nu \neq 0$

$$x = x_0 + \nu t - \nu\psi(x) \tag{11.14}$$

Since (11.14) is the integral of system (11.10) having a solution for all $t \in R^1$, equation (11.14) is solvable for x and $x = x(\nu t)$ can be expressed in the form

$$x(\nu t) = x_0 + \nu t + \theta(\nu t) \tag{11.15}$$

For $\theta(\nu t)$ formula (11.14) yields the estimate

$$|\theta(\nu t)| \leqslant \nu |\psi(x)| \leqslant \pi \left| \frac{\nu}{y(x)} \right| \tag{11.16}$$

We prove that $\theta(\nu t)$ is a periodic function. To do this, we introduce the substitution of the variable

$$x = x_0 + \nu t + \theta \tag{11.17}$$

and go over from equation (11.10) to the equation

$$\frac{d\theta}{dt} = y(x_0 + \nu t + \theta) - \nu \tag{11.18}$$

It is necessary to show that the solution of equation (11.18) $\theta = \theta(\nu t, 0)$ passing at $t = 0$ through the point 0 is periodic. Note first of all that if this equation has at least one periodic solution,

then all its solutions are periodic. The general solution of equation (11.10) then has the form

$$x = x_0 + \nu t + c + \theta(\nu t + c, \theta_0) \tag{11.19}$$

where c is an arbitrary constant, $-\infty < c < \infty$. By (11.17), the periodicity of the function $c + \theta(\nu t + c, \theta_0)$ in νt yields the periodicity of the general solution of equation (11.18).

Thus to prove the periodicity of the function $\theta = \theta(\nu t, 0)$, one must prove the existence of a periodic solution of equation (11.18). Since according to relations (11.16), (11.17) equation (11.18) has a bounded solution, taking into account the well-known results of [110], it has a periodic solution of period 2π with respect to νt. And in virtue of the property indicated above, all solutions of (11.18), and hence the function $\theta(\nu t)$, are periodic of period 2π with respect to νt. This completes the proof of Theorem 11.2.

We now turn to the problem of the existence and calculation of rotary motions. We shall start from the equation of the form

$$\frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} - g\left(x, \frac{dx}{dt}\right) = 0 \tag{11.20}$$

where λ is a parameter, and $g(x, y)$ is a function satisfying the same conditions as does the function $f(x, y)$.

According to Theorems 11.1 and 11.2 the rotary motions of system (11.20) are determined uniquely by the periodic solutions of period $2\pi/k$ of the equation

$$\frac{dy}{dx} = -\lambda + \frac{g(x, y)}{y} \tag{11.21}$$

considered in the strip

$$a \leqslant y \leqslant b, \qquad a > 0 \tag{11.22}$$

We find them using the numerical-analytic method. We impose additional conditions on $g(x, dx/dt)$: we consider that $g(x, y)/y = f_1(x, y)$ satisfies the inequalities

$$|f_1(x, y)| \leqslant M_1, \quad |f_1(x, y') - f_1(x, y'')| \leqslant K|y' - y''| \tag{11.23}$$

and moreover

$$M_1 \leqslant \frac{b-a}{2\pi}, \qquad K < \frac{1}{2} \tag{11.24}$$

i. e. (11.21) is a 2π-system in the domain (11.22).

Under these assumptions the sequence of periodic functions

$$y_m(x, x_0, y_0) = y_0 + \int_{x_0}^{x} (f_1(s,\ y_{m-1}(s, x_0\ y_0)) - \overline{f_1(s, y_{m-1}(s, x_0, y_0))})\, ds \tag{11.25}$$

converges uniformly to the periodic function $y(x)$, and the sequence $\Delta_m(x_0, y_0)$ converges to the Δ-constant of the point (x_0, y_0) once $(x_0, y_0) \in R^1 \times D_{f_1}$

$$D_{f_1}: \quad [a + M_1\pi,\ b - M_1\pi] \tag{11.26}$$

We put

$$q = 2K, \qquad \varepsilon_m = \frac{\pi}{3} M_1 \left(\frac{1}{1-q} + \sum_{j=1}^{3} \delta_{jm} \mathbf{B}_j \right) q^m \tag{11.27}$$

$$\lambda'_m = \min_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta_m(x_0, y_0)$$

$$\lambda''_m = \max_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta_m(x_0, y_0) \tag{11.28}$$

where $\mathbf{B}_j$ and δ_{jm} are calculated from formula (4.5) at $T = 2\pi$. We have

Theorem 11.3. *Let the equation*

$$\frac{dy}{dx} = \frac{g(x, y)}{y} \tag{11.29}$$

satisfy conditions (11.23) *and* (11.24), *i. e. be a* 2π*-system in the domain* (11.22). *Then*

(1) *There exist constants* λ' *and* λ'', *such that for all* λ *in the interval*

$$\lambda' \leqslant \lambda \leqslant \lambda'' \tag{11.30}$$

system (11.20) *has rotary motions.*

(2) *The inequalities*

$$\lambda' \leqslant \lambda'_m + \varepsilon_m q \leqslant \lambda''_m - \varepsilon_m q \leqslant \lambda'' \tag{11.31}$$

hold provided

$$\lambda''_m - \lambda'_m \geqslant 2\varepsilon_m q \tag{11.32}$$

(3) *The motion of system* (11.20) *passing through the point* $(x_0, y_0) \in R^1 \times D_{f_1}$ *is rotary if and only if*

$$\lambda = \Delta(x_0, y_0) \qquad (\Delta(x_0, y_0) = \lim_{m \to \infty} \Delta_m(x_0, y_0)) \tag{11.33}$$

(4) *The rotation number* $2\pi\nu$ *of the rotary motion* $x(t)$ *of system* (11.20) *passing at* $t = 0$ *through the point* $(x_0, y_0) \in R^1 \times D_{f_1}$ *satisfies the inequality*

$$|\nu - \nu_m| \leqslant 3\varepsilon_m \left(1 + \frac{M_1 \pi}{a}\right)^2 \tag{11.34}$$

where

$$\nu_m = \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{dx}{y_m(x, x_0, y_0)}\right)^{-1} \tag{11.35}$$

Proof. (1) Since equation (11.29) is a 2π-system in the domain (11.22), according to Theorem 5.1 for each $(x_0, y_0) \in R^1 \times D_{f_1}$ there is a Δ-constant. Put

$$\lambda' = \min_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta(x_0, y_0), \quad \lambda'' = \max_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta(x_0, y_0) \tag{11.36}$$

It follows from the continuity of $\Delta(x_0, y_0)$ in x_0, y_0 that for each λ in the interval $[\lambda', \lambda'']$ there exists a point $(\bar{x}, \bar{y}) \in R^1 \times D_{f_1}$ such that $\lambda = \Delta(\bar{x}, \bar{y})$. But $\Delta(\bar{x}, \bar{y})$ is the Δ-constant of a point $(\bar{x}, \bar{y})$ with respect to the 2π-system (11.29). Therefore, by the definition of the Δ-constant, among the solutions of equation (11.21) for the phase trajectories there is a periodic one of period 2π for any such λ.

(2) According to formulas (11.28) and (11.36) and inequality (7.8)

$$|\Delta(x_0, y_0) - \Delta_m(x_0, y_0)| \leqslant \varepsilon_m q \tag{11.37}$$

we have

$$\lambda' = \min_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta(x_0, y_0)$$

$$= \min_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta_m(x_0, y_0)$$

$$+ \left[\min_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta(x_0, y_0) - \min_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta_m(x_0, y_0) \right]$$

$$\leqslant \lambda'_m + \varepsilon_m q$$

$$\lambda'' = \max_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta(x_0, y_0)$$

$$= \max_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta_m(x_0, y_0)$$

$$- \left[\max_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta_m(x_0, y_0) \right.$$

$$\left. - \max_{(x_0, y_0) \in R^1 \times D_{f_1}} \Delta(x_0, y_0) \right]$$

$$\geqslant \lambda''_m - \varepsilon_m q$$

from which statement (2) follows.

(3) Now let the rotary motion of system (11.20) pass through the point $(x_0, y_0) \in R^1 \times D_{f_1}$. Then, by Theorem 11.2, the equation for phase trajectories has a periodic solution passing through the same point. System (11.29) being a 2π-system, the point (x_0, y_0) has a Δ-constant and, from its definition, the solution of equation (11.21) $y(x, x_0)$, $y(x_0, x_0) = y_0$, is periodic when $\lambda = \Delta$. The statement of the necessity theorem is established, and the sufficiency of condition (11.33) follows from the definition of the Δ-constant.

(4) The rotary motion $x(t)$ of system (11.20) passing through the point $(x_0, y_0) \in R^1 \times D_{f_1}$ takes place when $\lambda = \Delta(x_0, y_0)$, in virtue of property (3). The phase trajectory $y = y(x)$ is, by Theorem 11.2, a periodic function and can be found as the sequence of the periodic functions $y_m(x, x_0, y_0)$, the following estimate being valid by (4.4):

$$|y(x) - y_m(x, x_0, y_0)| \leqslant 3\varepsilon_m \tag{11.38}$$

The function $y_m(x, x_0, y_0)$ converging uniformly to the limiting function $y(x)$, the sequence $\{\nu_m\}$, by the definition of ν and ν_m, tends to the limiting value, ν, and the following estimate is valid:

$$|\nu - \nu_m| \leqslant 2\pi \frac{\int_0^{2\pi} \frac{|y(x) - y_m(x, x_0, y_0)|}{y(x)\, y_m(x, x_0, y_0)}\, dx}{\int_0^{2\pi} \frac{dx}{y(x)} \int_0^{2\pi} \frac{dx}{y_m(x, x_0, y_0)}}$$

$$\leqslant 3\varepsilon_m \frac{y(x_1)\, y_m(x_2, x_0, y_0)}{y(x_3)\, y_m(x_3, x_0, y_0)}$$

where $0 < x_1, x_2, x_3 < 2\pi$.

According to inequality (11.38) we have

$$y(x_1) \leqslant y(x_3) + M_1\pi$$

$$y_m(x_2, x_0, y_0) \leqslant y_m(x_3, x_0, y_0) + M_1\pi$$

hence

$$\frac{y(x_1)\, y_m(x_2, x_0, y_0)}{y(x_3)\, y_m(x_3, x_0, y_0)} \leqslant \left(1 + \frac{M_1\pi}{y(x_3)}\right)\left(1 + \frac{M_1\pi}{y_m(x_3, x_0, y_0)}\right)$$

$$\leqslant \left(1 + \frac{M_1\pi}{a}\right)^2$$

Therefore

$$|\nu - \nu_m| \leqslant 3\varepsilon_m(1 + M_1\pi/a)^2$$

which was to be established. Thus Theorem 11.3 is proved. It defines the values of λ at which system (11.20) has rotary motions. Here is an example to illustrate the effectiveness of this definition.

EXAMPLE 11.1. Let us consider a pendulum with linear friction and a constant rotational moment. The equation for its motion [4] is

$$\frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} + \sin x = \mu \tag{11.39}$$

where λ and μ are positive constants.

The equation of phase trajectories is

$$\frac{dy}{dx} = -\lambda + \frac{\mu - \sin x}{y} \tag{11.40}$$

Conditions (11.23) and (11.24) hold if $a^2 \geqslant 2(\mu + 1)\pi/3$, $b > a + 2\pi(\mu + 1)/a$. Under these conditions there exist rotary motions for all λ in the interval

$$\frac{(\mu + 1)^{1/2}}{\gamma^5} \leqslant \lambda \leqslant \frac{(3/2\pi)^{1/2}\mu\gamma^6 + (\mu + 1)(\gamma^2 + 3/2)}{(\mu + 1)^{1/2}(\gamma^2 + 3/2)\gamma^5}$$

if $\gamma = (3/2\pi)^{1/2}\ a(\mu + 1)^{-1/2}$ is such that

$$\frac{\mu}{\mu + 1} \geqslant \left(\frac{2\pi}{3}\right)^{1/2} \frac{\gamma^2 + 3/2}{\gamma^6}\ . \tag{11.41}$$

These solutions lie in the domain $y \geqslant (2\pi/3)^{1/2}\gamma(\mu + 1)^{1/2}$. To a zeroth approximation, $\lambda \approx \lambda_0 = \mu/y_0$, from which it follows that the rotary motion of system (11.39) is unique and passes through the point $y_0 = \mu/\lambda$ when $x = 0$.

It should be noted that the conclusions drawn from using the zeroth approximation are in good agreement with those of [4] and supplement them.

On the basis of Theorems 11.2 and 11.3 the calculation of rotary motion reduces to finding the function $y(x)$, the rotation number $2\pi\nu$ and the function $\theta(\nu t)$. The function $y(x)$, i. e. the periodic solution of equation (11.21), passing through the point (x_0, y_0) will be found as the limit of the sequence of periodic functions (11.25). The rotation number $2\pi\nu$ is defined as the limit of the sequence of numbers (11.35) and the function $\theta(\nu t)$ as the limit of the sequence of the functions

$$\frac{d\theta_m}{dt} = y_m(x + \nu_m t + \theta_{m-1}) - \overline{y_m(x_0 + \nu_m t + \psi_{m-1})} \tag{11.42}$$

$$\theta_0 = 0$$

Then $x(\nu t)$ is defined as the limit of the sequence $x_m(\nu t)$ where

$$x_m(\nu t) = x_0 + \nu_m t + \theta_m(\nu t) \tag{11.43}$$

Here is an illustration of the finding of rotary motion [4].

EXAMPLE 11.2. [4]. A heavy point of mass m is moving along a circle with a radius a, lying in a vertical plane, and rotating with an angular speed ω about the vertical diameter.

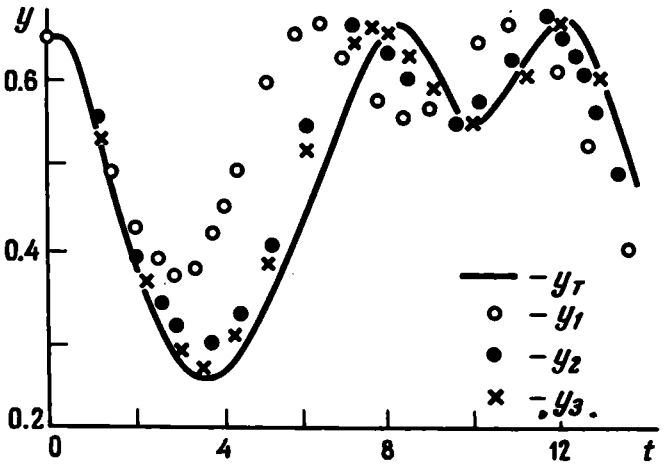

Fig. 2

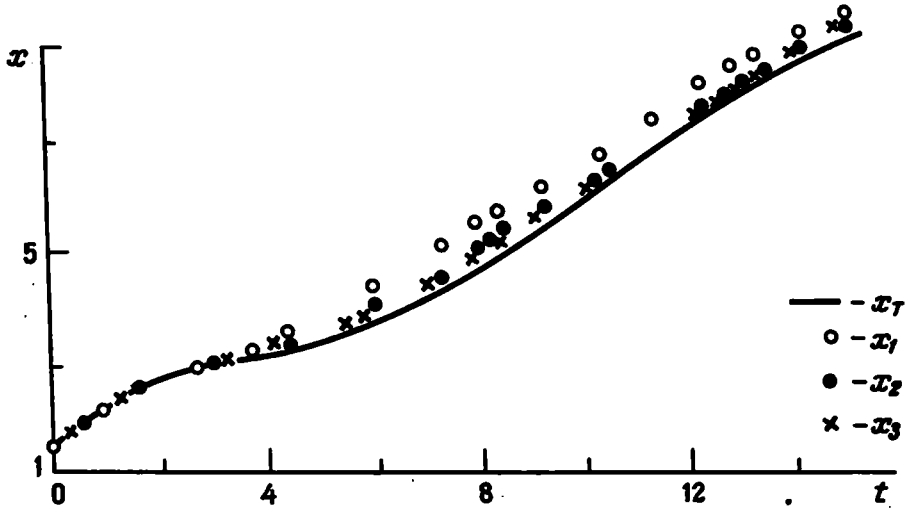

Fig. 3

If we neglect the forces of friction, the equation of motion of the point has the form

$$\frac{d^2x}{dt^2} + \omega^2 \sin x(\cos x - \mu) = 0 \qquad (11.44)$$

where $\mu = g/(a\omega^2)$, $m = 1$.

The equation of the phase trajectories in the domain $a \leqslant y \leqslant b$, where $a^2 \geqslant \pi\omega^2(1 + 2\mu)/3, b > a + \pi\omega^2(1 + 2\mu)/a$, satisfies equations (11.23), (11.24) and for $\lambda = 0$, system (11.44) has rotary motions which were calculated for the following values of parameters

TABLE 1

Approximations	ν_T	ν_1	ν_2	ν_3	T_T	T_1	T_2	T_3
1	3.1182	3.1198	3.1175		2.0150	2.0140	2.0155	
2	13.9970	13.9800			0.4489	0.4494		
3	0.4956	0.5405	0.5025	0.5012	12.6800	11.5876	12.5049	12.5356

and initial conditions:

$$(1)\ \omega = 1,\ \mu = \frac{1}{4},\ x_0 = \frac{\pi}{2},\ y_0 = 3.2$$

$$(2)\ \omega = 1,\ \mu = 4,\ x_0 = \frac{\pi}{2},\ y_0 = 14$$

$$(3)\ \omega = \frac{1}{2},\ \mu = \frac{1}{4},\ x_0 = \frac{\pi}{2},\ y_0 = 0.655$$

In the first two cases the first approximations for a period differ from the solutions obtained using the INTEGRAL-1 computer at most by 0.03 per cent. Figures 2 and 3 give the deviations of the first three approximations from the solutions obtained for the third case.

Table 1 gives approximate values of the rotation numbers $2\pi\nu$ and those of the periods, which were obtained from the first three approximations, ν_i, $T_i = 2\pi/\nu_i$, $i = 1, 2, 3$, as well as using the computer, ν_T and T_T.

12. ROTARY-OSCILLATORY MOTIONS OF A DOUBLE PENDULUM WITH A VIBRANT POINT OF SUSPENSION

In the previous section rotary motions of an autonomous system with one degree of freedom were discussed. Suppose now that we are given a system with n degrees of freedom and that its equation of motion has the form

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right) \tag{12.1}$$

where $x, f \in E_n$, $f(t, x, dx/dt)$ is a continuous function periodic in t and $x_1, x_2, \ldots, x_r$ $(1 \leqslant r \leqslant n)$ of periods $T, T_1, T_2, \ldots, T_r$ respectively.

For such a system, besides harmonic oscillations one can seek motions characterized by solutions of the form

$$x_i = \nu_i t + \Phi_i(t), \qquad i = 1, 2, \ldots, r$$

$$x_i = \Phi_i(t), \qquad i = r+1, \ldots, n \tag{12.2}$$

where $\Phi_i(t)$, $i = 1, 2, \ldots, n$, are periodic functions of period T in t, and $\nu_i = (T_i/T)k_i$ (k_i are positive integers).

It is natural to call such motions of a system rotary-oscillatory motions of period T. To find them it is sufficient to use new variables in the system of equations (12.1), namely

$$x_i = \nu_i t + z_i, \qquad i = 1, 2, \ldots, r \tag{12.3}$$

and to construct periodic solutions of period T of the system obtained. The latter problem is a problem of periodic solutions of a system of second-order differential equations with a right-hand side periodic in t of period T.

In the case when in the domain D

$$D: \quad x_i \in E_1, \qquad i = 1, 2, \ldots, r$$

$$(x_{r+1}, \ldots, x_n) \in D_1, \qquad \left(y_1 = \frac{dx_1}{dt}, \ldots, y_r = \frac{dx_r}{dt}\right) \in D_2$$

$$\left(y_{r+1} = \frac{dx_{r+1}}{dt}, \ldots, y_n = \frac{dx_n}{dt}\right) \in D_3 \tag{12.4}$$

where D_1, D_2, D_3 are closed bounded domains in the spaces E_{n-r}, E_r, ν_1, $\nu_2, \ldots, \nu_r \in D_2$, respectively, and the system of equations (12.1) is a T-system, the resulting problem can be solved using the numerical-analytic method.

By way of illustration let us consider a study of rotary-oscillatory motions of a double pendulum with a vibrant point of suspension [183]. The differential equations describing the motions of such a pendulum on the assumption that the point of suspension vibrates according to the law

$$x = a \sin \omega t \tag{12.5}$$

are of the form [185]

$$\frac{d^2\varphi_1}{dt^2} = \alpha_1 \left\{ - l_1 m_2 \left(\frac{d\varphi_1}{dt}\right)^2 \sin(\varphi_1 - \varphi_2)\cos(\varphi_1 - \varphi_2) \right.$$

$$- l_2 m_2 \left(\frac{d\varphi_2}{dt}\right)^2 \sin(\varphi_1 - \varphi_2)$$

$$- a\omega^2 \sin \omega t \left[m_2 \sin \varphi_2 \cos(\varphi_1 - \varphi_2)\right.$$

$$\left. \left. - (m_1 + m_2)\sin \varphi_1\right]\right.$$

$$
\begin{aligned}
&- \beta \frac{d\varphi_1}{dt} l_1 [m_1 + m_2 \sin^2(\varphi_1 - \varphi_2)] \\
&+ g m_2 \cos(\varphi_1 - \varphi_2) \sin \varphi_2 \\
&- g(m_1 + m_2) \sin \varphi_1 \Big\} \qquad (12.6)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d^2\varphi_2}{dt^2} = \alpha_2 \Big\{ & l_1 (m_1 + m_2) \left(\frac{d\varphi_1}{dt} \right)^2 \sin(\varphi_1 - \varphi_2) \\
&+ l_2 m_2 \left(\frac{d\varphi_2}{dt} \right)^2 \sin(\varphi_1 - \varphi_2) \cos(\varphi_1 - \varphi_2) \\
&- \beta l_2 \frac{d\varphi_2}{dt} [m_1 + m_2 \sin^2(\varphi_1 - \varphi_2)] \\
&- a\omega^2 \sin \omega t (m_1 + m_2) \\
&\times [\sin \varphi_1 \cos(\varphi_1 - \varphi_2) - \sin \varphi_2] \\
&+ g(m_1 + m_2) [\cos(\varphi_1 - \varphi_2) \sin \varphi_1 - \sin \varphi_2] \Big\}
\end{aligned}
$$

where φ_1, φ_2 are the angular variables characterized by the deviations of the first and the second pendulum from the vertical; m_1 and m_2 are the masses; l_1 and l_2 are the pendulum lengths;

$$\alpha_1 = \frac{1}{l_1 [m_1 + m_2 \sin^2(\varphi_1 - \varphi_2)]}$$

$$\alpha_2 = \frac{1}{l_2 [m_1 + m_2 \sin^2(\varphi_1 - \varphi_2)]}$$

β is the resistance coefficient; and g is the acceleration of gravity.

Let us consider a motion of the pendulum when the first pendulum with a mass $m_1 = 1$ and a length $l_1 = 1$ oscillates about its equilibrium position and the second one connected with it rotates around the suspension point with a vibration frequency ω.

According to what was stated above, we change variables in the system of equations (12.6) putting

$$\varphi_1 = \psi_1, \qquad \varphi_2 = \omega t + \psi_2 \qquad (12.7)$$

We obtain

$$\frac{d^2\psi_1}{d\tau^2} = \alpha_1 \left\{ -\frac{1}{2} m_2 [\sin 2\tau + \sin 2(\psi_1 - \psi_2)] \left(\frac{d\psi_1}{d\tau}\right)^2 \right.$$

$$- l_2 m_2 \left(1 + 2\frac{d\psi_2}{d\tau} + \left(\frac{d\psi_2}{d\tau}\right)^2\right)$$

$$\times [\sin(\psi_1 - \psi_2)\cos\tau - \cos(\psi_1 - \psi_2)\sin\tau]$$

$$- a m_2 [\cos(2\psi_2 - \psi_1)(\cos\tau - \cos^3\tau)$$

$$+ \sin\tau \sin\psi_2 \cos(\psi_1 - \psi_2)$$

$$- \sin^3\tau \sin(2\psi_2 - \psi_1)] + a(1 + m_2)$$

$$\times \sin\psi_1 \sin\tau - \frac{\beta}{\omega}\frac{d\psi_1}{d\tau} - \beta\frac{m_2}{2\omega}\frac{d\psi_1}{d\tau}$$

$$+ \beta\frac{m_2}{2\omega}\frac{d\psi_1}{d\tau}[\cos 2\tau \cos 2(\psi_1 - \psi_2)$$

$$+ \sin 2\tau \sin 2(\psi_1 - \psi_2)]$$

$$+ \frac{g m_2}{\omega^2}\left[\frac{1}{2}\sin 2\tau \cos(2\psi_2 - \psi_1)\right.$$

$$+ \sin^2\tau \sin(\psi_1 - \psi_2)\cos\psi_2$$

$$\left. + \cos^2\tau \sin\psi_2 \cos(\psi_1 - \psi_2)\right]$$

$$\left. - \frac{g(1 + m_2)}{\omega^2}\sin\psi_1 \right\} = f_1\left(\tau, \psi, \frac{d\psi}{dt}\right) \tag{12.8}$$

$$\frac{d^2\psi_2}{d\tau^2} = \alpha_2 \left\{ (1 + m_2)\left(\frac{d\psi_1}{d\tau}\right)^2 \right.$$

$$\times [\sin(\psi_1 - \psi_2)\cos\tau - \cos(\psi_1 - \psi_2)\sin\tau]$$

$$+\frac{1}{2}m_2 l_2\left(1+2\frac{d\psi_2}{d\tau}+\left(\frac{d\psi_2}{d\tau}\right)^2\right)$$

$$\times[\sin 2(\psi_1-\psi_2)+\sin 2\tau]-\frac{\beta l_2}{\omega}\frac{d\psi_2}{d\tau}$$

$$\times\left[1+\frac{1}{2}m_2(1-\cos 2\tau\cos 2(\psi_1-\psi_2)\right.$$

$$\left.-\sin 2\tau\sin 2(\psi_1-\psi_2)\right]-\frac{1}{2}a(1+m_2)$$

$$\times\left[\sin\psi_1\cos(\psi_1-\psi_2)\sin 2\tau\right.$$

$$+2\sin\psi_1\sin(\psi_1-\psi_2)\sin^2\tau$$

$$\left.-\frac{1}{2}\sin\psi_2\sin 2\tau-\cos\psi_2\sin^2\tau\right]$$

$$+\frac{g(1+m_2)}{\omega^2}[\cos(\psi_1-\psi_2)\sin\psi_1\cos\tau$$

$$+\sin(\psi_1-\psi_2)\sin\psi_1\sin\tau$$

$$\left.-\sin\tau\cos\psi_2-\cos\tau\sin\psi_2]\right\}$$

$$=f_2\left(\tau,\psi,\frac{d\psi}{dt}\right)$$

The right-hand sides of the system of equations (12.8) are seen to be defined in the domain

$$(\psi_1,\psi_2)\in E_2,\qquad \left|\frac{d\psi_1}{d\tau}\right|\leqslant d_1,\qquad \left|\frac{d\psi_2}{d\tau}\right|\leqslant d_2 \tag{12.9}$$

periodic of period $T=2\pi$ in τ and to satisfy inequalities of the form (10.50) with $\mathbf{M}=(M_1, M_2)$

$$\mathbf{K}_1=\begin{pmatrix}K_{11}^1 & K_{12}^1\\ K_{21}^1 & K_{22}^1\end{pmatrix},\qquad \mathbf{K}_2=\begin{pmatrix}K_{11}^2 & K_{12}^2\\ K_{21}^2 & K_{22}^2\end{pmatrix}$$

where

$$M_1 = \Big\{ m_2 d_1^2 + 2 l_2 m_2 (1 + 2 d_2 + d_2^2) + 4 a m_2 + a(1 + m_2) + \frac{\beta d_1}{\omega} + \frac{\beta d_1 m_2}{2\omega} + \frac{\beta d_1 m_2}{\omega} + \frac{3 g m_2}{2\omega^2} + \frac{g(1 + m_2)}{\omega^2} \Big\} \tag{12.10}$$

$$M_2 = \frac{1}{l_2} \Big\{ 2(1 + m_2) d_1^2 + m_2 l_2 (1 + 2 d_2 + d_2^2) + \frac{1}{2\omega} \beta l_2 (2 + 3 m_2) + \frac{1}{2\omega} \beta l_2 d_2 (2 + 3 m_2) - 3 a (1 + m_2) + \frac{1}{\omega_2} 4 g (1 + m_2) \Big\}$$

$$K_{ij}^1 = \max_{\tau, \psi, \frac{d\psi}{d\tau}} \left| \frac{\partial f_i}{\partial \psi_j} \right|, \quad K_{ij}^2 = \max_{\tau, \psi, \frac{d\psi}{d\tau} = \theta} \left| \frac{\partial f_i}{\partial \theta} \right| \tag{12.11}$$

$$i, j = 1, 2$$

are quite definite constants.

Suppose for the time being that the parameters m_2, l_2, β, a, ω are such that the system of differential equations (12.8) is a 2π-system in the domain (12.9). Then its approximate periodic solutions are functions of a sequence (10.10) and initial values of these solutions are determined approximately by the zeros of a function (10.37).

In a first approximation we have

$$\psi_1(\tau) = \psi_1^0 + \Big\{ - l_2 m_2 [- \sin(\psi_1^0 - \psi_2^0) \cos\tau + \cos(\psi_1^0 - \psi_2^0) \cos\tau + \cos(\psi_1^0 - \psi_2^0) \sin\tau] - a m_2 \Big[\cos(2\psi_2^0 - \psi_1^0) \Big(- \cos\tau + \frac{1}{3} \cos^3\tau \Big)$$

$$- \sin\tau \sin\psi_2^0 \cos(\psi_1^0 - \psi_2^0) - \sin(2\psi_2^0 - \psi_1^0)$$

$$\times \left(-\frac{2}{3}\sin\tau - \frac{1}{3}\sin^3\tau \right) \Bigg] - a(1 + m_2)\sin\psi_1^0 \sin\tau$$

$$+ \frac{gm_2}{\omega^2}\Bigg[-\frac{1}{8}\sin 2\tau \cos(2\psi_2^0 - \psi_1^0)$$

$$+ \frac{1}{8}\cos 2\tau \sin(\psi_1^0 - \psi_2^0)\cos\psi_2^0$$

$$- \frac{1}{8}\cos 2\tau \sin\psi_2^0 \cos(\psi_1^0 - \psi_2^0)\Bigg]\Bigg\} \qquad (12.$$

$$\psi_2(\tau) = \psi_2^0 + \frac{1}{l_2}\Bigg\{ -\frac{1}{8} m_2 l_2 \sin 2\tau$$

$$- \frac{1}{8\omega}\beta l_2 m_2 [\cos\tau \cos 2(\psi_1^0 - \psi_2^0)$$

$$- \sin 2\tau \sin 2(\psi_1^0 - \psi_2^0)] - \frac{a(1 + m_2)}{8}$$

$$\times [\cos 2\tau \sin\psi_1^0 \cos(\psi_1^0 - \psi_2^0)$$

$$+ \cos 2\tau \sin\psi_1^0 \sin(\psi_1^0 - \psi_2^0)$$

$$+ \sin 2\tau \sin\psi_2^0 - \cos\psi_2^0 \cos 2\tau]$$

$$+ \frac{g(1 + m_2)}{\omega_2}[- \sin\psi_1^0 \cos(\psi_1^0 - \psi_2^0)$$

$$\times \cos\tau - \sin(\psi_1^0 - \psi_2^0)\sin\psi_1^0 \sin\tau$$

$$+ \sin\tau \cos\psi_2^0 + \cos\tau \sin\psi_2^0]\Bigg\}$$

where ψ_1^0 and ψ_2^0 are the roots of the system of equations;

$$\left[\frac{gm_2}{\omega^2} - \frac{g(1+m_2)}{2\omega^2}\right]\sin\psi_1^0 = 0 \tag{12.13}$$

$$m_2 l_2 \sin 2(\psi_1^0 - \psi_2^0) - a(1+m_2)$$

$$\times [\sin\psi_1^0 \sin(\psi_1^0 - \psi_2^0) - \cos\psi_2^0] - \frac{\beta l_2}{\omega}(2+m_2) = 0$$

We find from the first equation of system (12.13)

$$\psi_1^0 = 2k\pi, \qquad k = 0, 1, 2, \ldots \tag{12.14}$$

and, substituting this value into the second equation, for the determination of ψ_2^0 have

$$\frac{a\cos\psi_2^0}{4} - \frac{m_2 l_2 \sin 2\psi_2^0}{2} - \frac{\beta l_2}{\omega} = 0 \tag{12.15}$$

Let us analyze the solution of this equation. Re-write equation (12.15) in the form

$$\cos\psi_2(\sin\psi_2 - k_1) = -k_2, \qquad \left(k_1 = \frac{a}{4l_2 m_2}, \quad k_2 = \frac{\beta}{m_2\omega}\right) \tag{12.16}$$

or, putting $\cos\psi_2 = z$, in the form

$$z^2(1-z^2) = (k_1 z - k_2)^2 \tag{12.17}$$

An analysis of the Sturm scheme for equation (12.17) shows that this algebraic equation has no real roots when

$$\beta > \frac{a\omega}{4l_2} \tag{12.18}$$

has at least two real roots when

$$\beta < \frac{a\omega}{4l_2} \tag{12.19}$$

and four real roots when

$$\beta = \frac{a\omega}{4l_2} \tag{12.20}$$

if we put

$$\frac{4l_2 m_2}{2^{1/2}a} = 1 \tag{12.21}$$

Thus in a first approximation the double pendulum with a vibrant point of suspension we are considering has, for the initial velocities in the domain (12.9), no rotary-oscillatory motion with the vibration frequency when inequality (12.18) holds; has at least two such motions when $\beta < a\omega/(4l_2)$ and four such motions when inequalities (12.20) and (12.21) hold.

Formulas (12.12) determine approximate periodic solutions which do in fact exist in the system of equations (12.8) when ψ_1^0 and ψ_2^0 are approximate values of the zeros of the Δ-constant of the 2π-system (12.8). We show that this is the case if one assumes that the mass m_2 of the second pendulum, the oscillation amplitude a and the oscillation period $T = 2\pi/\omega$ are sufficiently small,

Indeed, on this assumption the system of equations (12.8) is a 2π-system in the domain (12.9) when

$$d_1^2 < \frac{3\pi l_2}{10} \tag{12.22}$$

$$d_2 > \frac{2d_1^2}{l_2} \tag{12.23}$$

Then formulas (12.12) determine an approximate expression for periodic solutions of system (12.8), if, according to the requirements of Theorem 10.2, the roots ψ_1^0 and ψ_2^0 of equations (12.13) are simple and inequality (10.55) holds, i. e.

$$\left|\frac{g}{2\omega^2}(2+m_2)\sin\psi_1\right| + \left|\frac{1}{2}m_2 l_2 \sin 2(\psi_1-\psi_2)\right|$$

$$-\frac{1}{2}|a(1+m_2)[\sin\psi_1\sin(\psi_1-\psi_2)-\cos\psi_2]|$$

$$-\left|\frac{1}{2\omega}\beta l_2(2+m_2)\right| > M_2\left\{\pi\left(K_{12}^1K_{22}^1+\frac{5}{3\pi}K_{12}^2K_{22}^1\right)\right.$$

$$+\pi\left(K_{12}^1K_{21}^1+K_{22}^1K_{22}^1+\frac{5}{3\pi}K_{21}^2K_{12}^1+\frac{5}{3\pi}K_{22}^2K_{22}^1\right.$$

$$\left.\left.+\frac{5}{3\pi}K_{21}^1K_{12}^2+K_{22}^1K_{22}^1+K_{21}^2K_{12}^2\right)\right\} \tag{12.24}$$

on the boundary of a neighbourhood of theirs lying in D.

It is evident that, having in mind (12.10), the right-hand side of inequality (12.24) has the order of smallness

$$\min\left[\frac{m_2^{\alpha_1}a^{\alpha_2}}{\omega^{\alpha_3}}, d_1^3\right] \tag{12.25}$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 2, \qquad \alpha_i = 0,\ 1,\ 2, \qquad i = 1,\ 2,\ 3$$

So, putting d_1^3 equal to the minimum of the values $m_2^{\alpha_1}a^{\alpha_2}/\omega^{\alpha_3}$

$$d_1^3 = \min_{\alpha_1 + \alpha_2 + \alpha_3 = 2} \frac{m_2^{\alpha_1}a^{\alpha_2}}{\omega^{\alpha_3}} = \mu^2 \tag{12.26}$$

the left-hand side of inequality (12.24) will have an order of smallness lower by one than the right-hand side on the boundary of the domain of the form

$$|\psi_1| < \pi, \quad |\psi_2 - \psi_2^0| < \frac{r}{2} \tag{12.27}$$

where r is the minimum distance between the roots of the system of equations (12.13). This shows that all the conditions of the theorem of the existence of periodic solutions hold for the simple roots of system (12.13), at least in the domain

$$\psi_1,\ \psi_2 \in E_2$$

$$\left|\frac{d\psi_1}{d\tau}\right| \leqslant \mu^{2/3} \tag{12.28}$$

$$\left|\frac{d\psi_2}{d\tau}\right| \leqslant \frac{4}{l_2}\mu^{4/3}$$

Thus, a double pendulum whose suspension point vibrates by the law $x = a\sin\omega t$ with a small amplitude a and a high frequency ω performs rotary-oscillatory motions which in a first approximation have the form (12.7) where ψ_1 and ψ_2 are determined from formulas (12.12) and ψ_1^0 and ψ_2^0 are approximate initial values of the oscillatory motions, when, for example, $1 < 4l_2m_2/a$

$$\psi_1^0 = 0$$

$$\psi_2^0 = \left\{\frac{\pi}{2}, \frac{3}{2}\pi\right\}$$

and when $\dfrac{4l_2m_2}{a} = 1$

$$\psi_1^0 = 0$$

$$\psi_2^0 = \begin{cases} \dfrac{\pi}{2} \\ \dfrac{3}{2}\pi \\ \dfrac{\pi}{2} + 2k\pi \end{cases}$$

13. PERIODIC SOLUTIONS OF COUNTABLE SYSTEMS

As is seen from the definition, the conditions under which the system of differential equations (2.4) is a T-system do not directly comprise its size. This circumstance allows one to easily extend the abstract scheme of the numerical-analytic method to countable systems of differential equations and to obtain for these equations statements and estimates similar to those established earlier for equations in a finite-dimensional space.

Indeed, let $x = (x_1, x_2, \ldots, x_n, \ldots)$ be a point of a space $\mathbf{m}$ of a bounded numerical sequence with the norm

$$|x| = \sup_n |x_n| \tag{13.1}$$

and let $f(t, x) = (f_1(t, x), \ldots, f_n(t, x), \ldots)$ be a continuous function of the variables t, x changing in the domain

$$t \in R^1, \qquad x \in D \tag{13.2}$$

(D being a bounded closed domain of the space $\mathbf{m}$) and taking values in the space $\mathbf{m}$.

Consider a countable system of differential equations

$$\frac{dx}{dt} = f(t, x) \tag{13.3}$$

regarding $f(t, x)$ as a periodic function of the argument t of period T and satisfying the inequalities

$$|f(t, x)| \leqslant \mathbf{M} \tag{13.4}$$

$$|f(t, x') - f(t, x'')| \leqslant \mathbf{K}|x' - x''| \tag{13.5}$$

for all t, x, x', x'' in the domain

$$t \in R^1, \qquad x,\ x',\ x'' \in D \tag{13.6}$$

where $\mathbf{M} = (M_1, \ldots, M_n, \ldots)$, $\mathbf{K} = (K_{ij}, i, j = 1, 2, \ldots)$ are the infinite-dimensional vector and matrix with nonnegative elements, respectively.

By a solution of the system of equations (13.3) we mean as usual the function $x(t) = (x_1(t), \ldots, x_n(t), \ldots)$ defined for t in an interval (a, b), taking a value in a space $\mathbf{m}$, continuously differentiable and satisfying equation (13.3). Conditions which ensure that solutions of the system of equations (13.3) exist are natural and known from [136, 189].

It will be recalled [60, 61] that the matrix $\mathbf{K}$ generates the linear operator $\mathbf{K}$ in the space $\mathbf{m}$ when

$$\sup_i \sum_{j=1}^{\infty} K_{ij} < \infty \tag{13.7}$$

the norm of the operator $\mathbf{K}$ in the space $\mathbf{m}$ being determined by the left-hand side of inequality (13.7)

$$|\mathbf{K}| = \sup_i \sum_{j=1}^{\infty} K_{ij} \tag{13.8}$$

the operator $\mathbf{K}$ is said to be quite regular if

$$|\mathbf{K}| < 1 \tag{13.9}$$

The assumptions guaranteeing that the basic statements proved above for equations in a finite-dimensional space are valid for the system of equations (13.3), i. e. the conditions under which (13.3) is a T-system, are as follows:

(1) the domain D_f is nonempty

$$D_f \neq \varnothing \tag{13.10}$$

(2) the operator $\frac{T}{\pi}\mathbf{K}$ is quite regular

$$\frac{T}{\pi} \quad |\mathbf{K}| \leqslant q < 1 \tag{13.11}$$

The following statements hold:

Theorem 13.1. *Let the system of equations* (13.3) *be a T-system in the domain D. Then*

(1) *the sequence of the functions periodic in t, τ of period T*

$$x_m(t, \tau, x_0) = x_0 + \int_\tau^t f(s, x_{m-1}(s, \tau, x_0))\, ds$$

$$- \int_\tau^t \overline{f(s, x_{m-1}(s, \tau, x_0))}\, ds \tag{13.12}$$

converges uniformly with respect to

$$(t, \tau, x_0) \in R^1 \times R^1 \times D_f \tag{13.13}$$

to the function $x^0(t, \tau, x_0)$ *defined in the domain* (13.13) *and satisfying the equation*

$$x(t, \tau, x_0) = x_0 + \int_\tau^t f(s, x(s, \tau, x_0))\, ds$$

$$- \int_\tau^t \overline{f(s, x(s, \tau, x_0))}\, ds \tag{13.14}$$

and

$$|x^0(t, \tau, x_0) - x_m(t, \tau, x_0)|$$

$$\leqslant \frac{|\mathbf{M}|\, T}{2} q^m \left(\frac{1}{1-q} + \sum_{j=1}^{3} \delta_{jm} \beta_j \right) = \frac{|\mathbf{M}|\, T}{2} \rho_m$$

$$m = 1, 2, \ldots \tag{13.15}$$

β_j *are constants determined according to formulas* (4.5) *where the matrices E and* $\mathbf{K}$ *are replaced by the numbers* 1 *and* $|\mathbf{K}|$, *respectively*;

(2) *any point* $(\tau, x_0) \in R^1 \times D_f$ *has the* Δ*-constant determined by the equality*

$$\Delta(\tau, x_0) = \overline{f(t, x^0(t, \tau, x_0))} = \lim_{m \to \infty} \Delta_m(\tau, x_0)$$

$$= \lim_{m \to \infty} \overline{f(t, x_m(t, \tau, x_0))} \tag{13.16}$$

and satisfying the inequalities

$$|\Delta| \leqslant |\mathbf{M}|$$

$$|\Delta(\tau, x_0) - \Delta(\tau, x_0')|$$

$$\leqslant |\mathbf{K}|\left(1 + \frac{T}{3}|\mathbf{K}|\frac{1}{1-q} + \beta\right)|x_0 - x_0'|$$

$$= \sigma|x_0 - x_0'| \tag{13.17}$$

$$|\Delta(\tau, x_0) - \Delta_m(\tau, x_0)| \leqslant \frac{\pi}{3}|\mathbf{M}|\, q\rho_m,$$

$$m = 1, 2, \ldots \tag{13.18}$$

where β *is a constant equal to the value of expression* (5.12′) *in which the matrix* $\mathbf{K}$ *is replaced by the number* $|\mathbf{K}|$;

(3) *the solution* $x = x(t, \tau, x_0)$ *of the system of equations* (13.3) *taking at* $t = \tau$ *the value* $x_0 \in D_f$ *is periodic of period* T *if and only if the* Δ*-constant is zero at the point* (τ, x_0), *with*

$$x(t, \tau, x_0) = x^0(t, \tau, x_0) \tag{13.19}$$

(4) *if the system of equations* (13.3) *has the periodic solution* $x = x(t, \tau, x_0)$ and $x_0 \in D_1 \subset D_f$, *then for any* $m \geqslant 1$ *the following inequality is valid*

$$|\Delta_m(\tau, x_0)| < \sigma \max_{x \in D_1}|x - x_0| + \frac{T}{3}|\mathbf{M}|\,|\mathbf{K}|\,\rho_m \tag{13.20}$$

(5) *if* $f(t, x) = -f(-t, x)$ *for all* $t \in R^1$, $x \in D$, *then* $\Delta_m(\tau, x_0) \equiv \equiv 0$ *for all* $m = 0, 1, 2, \ldots, \tau \in R^1$, $x_0 \in D_f$.

The theorem is proved in the same way as the corresponding theorems for the finite-dimensional case and allows one to approximately construct a periodic solution of the countable T-system on the assumption that it exists and passes through a point of the domain D_f at $t = \tau$. In addition, it allows one to pose and solve the problem of periodic control for countable T-systems.

The problem of the existence of periodic solutions of period T for the system of equations (13.3) is solved by

Theorem 13.2. *Let the following conditions hold for a T-system* (13.3) *defined in the domain D of the space* **m**:

(a) *for a certain* $\tau \in R^1$ *and an integer m the map*

$$\Delta_m : \Delta_m = \Delta_m(x_0) = \overline{f(t, x_m(t, \tau, x_0))} \tag{13.21}$$

of the domain D_f *into* $\Delta_m D_f$

$$D_f \to \Delta_m D_f$$

has a singular point $x_0 = x^0$

$$\Delta_m(x^0) = 0 \tag{13.22}$$

(b) *there exists a closed bounded domain* D_1 *belonging to* D_f *and containing a point* x^0 *such that the operator* Δ_m *is a homeomorphism of* D_1 *onto* $\Delta_m D_1$;

(c) *the inequality*

$$\inf_{x \in \Gamma_{D_1}} |\Delta_m(x)| \geqslant \frac{\pi}{3} |\mathbf{M}| q \rho_m \tag{13.23}$$

holds on the boundary Γ_{D_1} *of the domain* D_1. *Then the system of equations* (13.3) *has a periodic solution of period T,* $x = x(t)$, *for which* $x(\tau) \in D_1$.

In the case $m = 0$, we see that $\Delta_0(x) = \overline{f(t, x)}$, Theorem 13.2 then establishes the existence of periodic solutions of a countable system of equations (13.3) on the basis of the analysis of the corresponding averaged system

$$\frac{d\xi}{dt} = \overline{f(t, \xi)} = \Delta_0(\xi) \tag{13.24}$$

Condition (13.23) of the theorem takes for this case the form

$$\inf_{x \in \Gamma_{D_1}} |\Delta_0(x)| \geqslant \frac{\pi}{3} |\mathbf{M}| q \tag{13.25}$$

We proceed to prove the theorem. Let us establish a subsidiary statement.

Lemma 13.1. *Let D be a closed bounded domain of the space* **m** *and* A_0 *and A continuous maps of D into* **m**, *with* $|Ax - A_0 x| \leqslant \varepsilon$. *Then, if* A_0 *is a homeomorphism, AD contains the set* $A_0 D \setminus \Pi(\varepsilon)$ *consisting of the points of the domain* $A_0 D$ *which are contained in* $A_0 D$ *together with their* ε*-neighbourhoods.*

Proof. Since A_0 is a homeomorphism, the continuous map AA_0^{-1}, which maps A_0D onto AD, is defined on A_0D. Putting $A_0D = D_1$ and $AA_0^{-1} = A_1$, we reduce the proof of the lemma to that for the case $A_0 = I$, where I is an identity map.

So, let $A_0 = I$. One must prove that $D\backslash\Pi(\varepsilon) \subset AD$. We take $x_0 \in D\backslash\Pi(\varepsilon)$; then the ball $T_\varepsilon(x_0)$: $|x - x_0| \leqslant \varepsilon$ is contained in D

$$T_\varepsilon(x_0) \subset D$$

We define the continuous operator

$$A^*x = x_0 + (x - Ax) \tag{13.26}$$

on $T_\varepsilon(x_0)$. We consider the operator as a transformation in some space C^∞, having defined the latter [189] as a space of numerical sequences $x = (x_1, \ldots, x_n, \ldots)$ uniformly bounded by the number ε, $|x_i| \leqslant \varepsilon$ and having the norm

$$|x| = \sum_{k=1}^{\infty} \frac{1}{2^k} |x_k| \quad . \tag{13.27}$$

When $x_0 = 0$, the ball $T_\varepsilon(x_0)$ coincides with C^∞ and the operator A^* transforms C^∞ onto itself

$$|A^*x - x_0| = |x - Ax| < \varepsilon$$

Moreover, the family of kth coordinates $v_k = \{A^*x\}_k$ of the transformation A^* is compact for any $k = 1, 2, \ldots$; from which, taking into account the theorem on the compactness of a set in a space C^∞ [189] follows the compactness of the set of points $\{v_k\}$ as the set of points of the space C^∞.

From the Schauder theorem [3] we now conclude that the operator A^* has a fixed point

$$A^*x^* = x^* \tag{13.28}$$

in C^∞.

It follows from (13.28) in conjunction with (13.26) that

$$Ax^* = x_0 \tag{13.29}$$

Equality (13.29) means that x_0 is an image of the point x^* in the map A. Since the equality holds for any point $x_0 \in D\backslash\Pi(\varepsilon)$ and x^* belongs to the ball $T_\varepsilon(x_0)$ and, therefore, to the set D, it also means that the whole set $D\backslash\Pi(\varepsilon)$ is an image of the set D in the map A. We have, therefore, the inclusion $D\backslash\Pi(\varepsilon) \subset AD$ which it was required to prove.

The case $x_0 \neq 0$ can be reduced to that considered above by translating the coordinate system.

Taking into account Lemma 13.1 and the estimate (13.18), the set $\Delta_m D_1 \setminus \prod\left(\frac{\pi}{3} |\mathbf{M}| q\rho_m\right)$ is contained in ΔD_1. If the set contains the origin of the coordinates of the coordinate system $\Delta = (\Delta^1, \ldots, \Delta^n, \ldots)$, then the set ΔD_1 also contains that origin. This is sufficient for the T-system to have a periodic solution of period T.

To prove Theorem 13.2 it is sufficient to show the relation

$$0 \in \Delta_m D_1 \setminus \prod\left(\frac{\pi}{3} |\mathbf{M}| q\rho_m\right)$$

Since the map $\Delta_m(x)$ is a homeomorphism, $\Delta_m D_1$ is a domain (see [3], for example) and the boundary $\Gamma_{\Delta_m D_1}$ of the domain $\Delta_m D_1$ is an image of the boundary Γ_{D_1} of the domain D_1

$$\Gamma_{\Delta_m D_1} = \Delta_m \Gamma_{D_1} \tag{13.30}$$

By condition (a) of Theorem 13.2 the set $\Delta_m D_1$ contains the zero point of the system $\Delta = (\Delta^1, \ldots, \Delta^n, \ldots)$. This point belongs to the set $\Delta D_1 \setminus \Pi(r)$ when all the points z of the boundary $\Gamma_{\Delta_m D_1}$ of the set $\Delta_m D_1$ are at least at a distance r away from it. It follows that zero belongs to the set $\Delta_m D_1 \setminus \prod\left(\frac{\pi}{3} |\mathbf{M}| q\rho_m\right)$ when

$$\inf_{z \in \Gamma_{\Delta_m D_1}} |z| \geqslant \frac{\pi}{3} |\mathbf{M}| q\rho_m \tag{13.31}$$

By equality (13.30) we have the identity

$$z = \Delta_m(x)|_{x \in \Gamma_{D_1}} \tag{13.32}$$

owing to which inequality (13.31) takes the form

$$\inf_{x \in \Gamma_{D_1}} |\Delta_m(x)| \geqslant \frac{\pi}{3} |\mathbf{M}| q\rho_m \tag{13.33}$$

It follows from the above that when (13.33) holds a T-system has a periodic solution of period T, which it was required to establish. Thus the theorem is proved.

14. EXTENSION OF NUMERICAL-ANALYTIC METHOD TO EQUATIONS WITH DEVIATING ARGUMENT, INTEGRO-DIFFERENTIAL AND PARTIAL DIFFERENTIAL EQUATIONS

The method of investigating periodic solutions discussed above was presented by A.M. Samoilenko in [162, 163], and the scheme of the method for second-order and countable systems of differential equations in [164, 165]. The investigation of rotary motions of autonomous systems is due to A.M. Samoilenko and A.Ya. Gadionenko [166] and that of rotary-oscillatory motions of a double pendulum with a vibrant point of suspension to T.G. Strizhak [183-185].

In this section we attempt a brief presentation of the results pertaining to the extension of the numerical-analytic method to various classes of differential equations, such as equations with deviating argument, integro-differential and partial differential equations.

Let us consider differential equations with deviating argument. The basic statements of the method were extended to them by A.M. Samoilenko and D.I. Martynyuk [107] and comprehensively developed by Martynyuk [102, 103] and Martynyuk in association with Kozubovskaya and Kharabovskaya [77, 106].

Note that application of the numerical-analytic method to equations with deviating argument allows one to find periodic solutions which satisfy the equation in the whole time axis. Moreover, it allows the deviation value to be included in the existence conditions of periodic solutions, thereby making it possible to follow the emergence and disappearance of periodic solutions. This is well illustrated by the example of the equation [107]

$$\frac{dx(t)}{dt} = \varepsilon [x^2(t) - \sin t - \cos t x(t-\tau) + 0.001] \qquad (14.1)$$

analyzed in the domain

$$|x| \leqslant 0.25 \qquad (14.2)$$

when

$$0 < \varepsilon \leqslant 0.06, \qquad 0 \leqslant \tau \leqslant 2\pi \qquad (14.3)$$

For such an equation, when ε are sufficiently small to lead to the inequality $(4 + \sin\tau)^2 \varepsilon^2 < 4(2 + \varepsilon^2)(0.002 + 3\varepsilon^2 - \varepsilon\cos\tau)$, there

exist no periodic solutions of period 2π at any $0 \leqslant \tau \leqslant 2\pi$. Such a solution does exist already when $\varepsilon = 0.01$ for all τ in the intervals

$$0 \leqslant \tau \leqslant \frac{5\pi}{18}, \qquad \frac{31}{18}\pi < \tau \leqslant 2\pi \tag{14.4}$$

The conditions separating a T-system from a system of equations with deviating argument

$$\frac{dx(t)}{dt} = f(t, x(t), x(t-\tau)) \tag{14.5}$$

are as follows: the function $f(t, x, y)$ is defined in the domain

$$t \in R^1, \quad x, y \in D \times D \tag{14.6}$$

continuous in the totality of variables, periodic in t of period $T (0 \leqslant \tau \leqslant T)$ and satisfies the inequalities

$$|f(t, x, y)| \leqslant \mathbf{M} \tag{14.7}$$

$$|f(t, x', y') - f(t, x'', y'')| \leqslant \mathbf{K}_1 |x' - x''| + \mathbf{K}_2 |y' - y''|$$

$$t, x, y, x', y', x'', y'' \in R^1 \times D \times D \tag{14.8}$$

the set D_f is nonempty and the eigenvalues of the matrix

$$\mathbf{Q} = (\mathbf{K}_1 + \mathbf{K}_2)\left[\frac{T}{3} + \frac{3}{2T} + \frac{3}{2T}\tau^2\left(1 - \frac{\tau}{T}\right)^2\right]$$

lie within the circle of the unit radius.

A T-system with deviating argument satisfies the basic statements of Theorem 13.1 and Theorem 7.1 after introducing corresponding changes. We shall not analyze these results any further here, neither shall we dwell on similar results for systems of second-order equations with deviating argument and countable systems because they are presented and discussed in detail in "Lectures on Theory of Oscillations of Systems with Delay" by Yu.A. Mitropolsky and D. I. Martynyuk [124].

G. Vakhabov [209-211] used the numerical-analytic method to study periodic solutions of integro-differential equations. Such a use does not present any principal difficulties in the case when the integral term of an integro-differential equation is a Fredholm

one or, in a more general case, has the form

$$\int_0^{h(t)} F(t, s, x(s))\, ds \tag{14.8'}$$

where $h(t)$ is a function periodic in t of period T.

Having considered a system of integro-differential equations of the form

$$\frac{dx}{dt} = f(t, x(t)) + \int_0^{h(t)} F(t, s, x(s))\, ds \tag{14.9}$$

and regarding $h(t), f(t, x), F(t, s, x)$ as a function periodic in t and s of period T, Vakhabov separated a class of T-systems and proved for them basic results similar to Theorems 13.1 and 7.1.

The application of the numerical-analytic method to the study of integro-differential equations whose integral term is a Volterra one is connected with certain difficulties. This is due to some rigidity of conditions necessary for the existence of periodic solutions of such systems. We shall discuss it in more detail.

Let a system of equations

$$\frac{dx}{dt} = f\left(t, x, \int_0^t F(t, s, x(s))\, ds\right) \tag{14.10}$$

be given, where $f(t, x, y) = (f_1(t, x, y), \ldots, f_n(t, x, y))$ and $F(t, s, x) = (F_1(t, s, x), \ldots, F_m(t, s, x))$ are continuous functions periodic in t and s of period T, defined in the domain

$$t, s \in R^1 \times R^1$$

$$x = (x_1, x_2, \ldots, x_n) \in D, \quad y = (y_1, y_2, \ldots, y_m) \in D^1 \tag{14.11}$$

D and D^1 are bounded domains of Euclidean spaces E_n and E_m.

Unlike ordinary differential equations, for which the function $x(t + T)$ is a solution as well as $x(t)$, in systems of the form (14.10) $x(t + T)$ may fail to be a solution. For periodic solutions of period T for the system of integro-differential equations (14.10) such a situation must not occur. This leads to

$$f\left(t, x(t), \int_0^t F(t, s, x(s))\, ds + \int_0^T F(t, s, x(s))\, ds\right)$$

$$\equiv f\left(t, x(t), \int_0^t F(t, s, x(s))\, ds\right) \tag{14.12}$$

for any solution periodic in t of period T of system (14.10).

Equality (14.12) is equivalent to the condition

$$\int_0^T F(t, s, x(s))\, ds = 0 \tag{14.13}$$

whenever for each fixed t, $x \in R^1 \times D$ the vector function $f(t, x, y)$ is a univalent map of the domain D^1 onto its image $f(D^1)$. In this case, for a solution $x(t)$ to be periodic, it must satisfy relation (14.13). This results in additional difficulties associated with satisfying relation (14.13), when using the numerical-analytic method to seek periodic solutions of a system of equations (14.10). Thus, to determine the conditions ensuring that periodic solutions of an integro-differential T-system (14.10) exist, besides the conditions similar to those of Theorem 7.1, one has to require that the relation

$$\int_0^T F(t, s, x_m(s, x_0))\, ds = 0 \tag{14.14}$$

should hold for all $x_0 \in D$.

The restrictions which separate T-systems from integro-differential equations under consideration as well as all constructions and foundations of the numerical-analytic scheme are, except for the peculiarity mentioned above, similar in the case of integro-differential T-systems to those for T-systems of ordinary differential equations.

The numerical-analytic method for partial differential equations has been developed by B.P. Tkach [191-194]. He has considered a problem of solutions periodic in time of a system of differential equations of hyperbolic type, both those with deviating argument and of neutral type. Unlike ordinary differential equations whose periodic solutions are determined by an initial value, i. e. by a point of finite-dimensional space, periodic solutions of partial differential equations are determined by an initial function and hence by a point of functional space. Owing to this, one must indicate the value of the initial function in the scheme of the numerical-analytic method as applied to partial differential equations even when constructing the very first iteration. This leads to a modification of the iteration process which manifests itself, as applied to the general scheme of the numerical-analytic method, in each iteration $x_m(t)$ being determined by the initial value

$$x_m(0) = x_m^0 = x_{m-1}^0 + \delta_m, \quad x_0 = \delta_0$$

which is the root of the equation

$$\Delta_m(x^0_{m-1} + \delta_m) = 0 \tag{14.15}$$

We shall consider in somewhat more detail the peculiarities of, and difficulties in extending the numerical-analytic method to partial differential equations.

The periodic solution problem studied in [127] is as follows. Given a system of differential equations of the form

$$\frac{\partial^2 u(t, x)}{\partial t\, \partial x} = f\left(t, x, u(t, x), u(t-\tau, x), \frac{\partial u}{\partial t}(t, x), \frac{\partial u}{\partial t}(t-\tau, x),\right.$$

$$\left.\frac{\partial u}{\partial x}(t, x), \frac{\partial u}{\partial x}(t-\tau, x), \frac{\partial^2 u}{\partial t\, \partial x}(t-\tau, x)\right) \tag{14.16}$$

where t and x are scalars, it is required that an initial function $v(x)$ should be chosen, given a function $u_0(t)$, such that the solution $u(t, x)$, which satisfies the conditions

$$u(0, x) = u_0(0) + v(x)$$

$$u(t, 0) = u_0(t) + v(0) \tag{14.17}$$

is periodic in t of period T, and the solution should be found. This naturally presupposes that the functions $f(t, x, u(t, x), \ldots)$ and $u_0(t)$ are periodic in t of period T. The problem is equivalent to choosing a function $v(x)$ which ensures that the solution of the system of integral equations

$$u(t, x) = u_0(t) + v(x) + \int_0^t \int_0^x f\left\{t, x, u(t, x), u(t-\tau, x)\right.$$

$$\frac{\partial u}{\partial t}(t, x), \frac{\partial u}{\partial t}(t-\tau, x), \frac{\partial u}{\partial x}(t, x),$$

$$\frac{\partial u}{\partial x}(t-\tau, x)$$

$$\left.\frac{\partial^2 u}{\partial t\, \partial x}(t-\tau, x)\right\} dx\, dt \tag{14.18}$$

is periodic in t of period T and that this solution can be constructed.

A periodic solution $u(t, x)$ is sought by the scheme

$$u_0(t, x) = u_0(t) + v_0(x)$$

. . . . . .

$$u_m(t, x) = u_0(t) + v_0(x) + \sum_{i=1}^{m} \delta_i(x) + \int_0^t \int_0^x f\Big\{t, x, u_{m-1}(t, x)$$

$$u_{m-1}(t-\tau, x), \frac{\partial u_{m-1}}{\partial t}(t, x), \frac{\partial u_{m-1}}{\partial t}(t-\tau, x)$$

$$\frac{\partial u_{m-1}}{\partial x}(t, x), \frac{\partial u_{m-1}}{\partial x}(t-\tau, x)$$

$$\frac{\partial^2 u_{m-1}}{\partial t\, \partial x}(t-\tau, x)\Big\}\, dx\, dt \tag{14.19}$$

and the functions $v_0(x)$ and $\delta_i(x)$ are determined from the equations

$$\frac{1}{T}\int_0^T f\Big\{t, x, u_0(t) + v_0(x), u_0(t-\tau) + v_0(x), \frac{du_0(t)}{dt}$$

$$\frac{du_0(t-\tau)}{dt}, \frac{dv_0(x)}{dx}; \frac{dv_0(x)}{dx}, 0\Big\}\, dt = 0 \tag{14.20}$$

$$\frac{1}{T}\int_0^T f\Big\{t, x, u_i(t, x), u_i(t-\tau, x), \frac{\partial u_i}{\partial t}(t, x), \frac{\partial u_i}{\partial t}(t-\tau, x)$$

$$\frac{\partial u_i}{\partial x}(t, x), \frac{\partial u_i}{\partial x}(t-\tau, x), \frac{\partial^2 u}{\partial t\, \partial x}(t-\tau, x)\Big\}\, dt \tag{14.21}$$

$$i = 1, 2, \ldots, m, \ldots$$

Here great possibilities open up if one solves the problem in a sufficiently small interval of the variable x. Thus, in particular, if the right-hand side of system (14.16) is independent of the derivatives of this variable, the iteration scheme of the numerical-analytic method (14.19), (14.21) could always be realized under the most natural assumptions concerning the function $f(t, x, u(t, x), \ldots)$.

As to the general form of system (14.16), difficulties in constructing periodic solutions arise from the fact that the function $v_0(x)$ has

itself to be sought as the solution of an ordinary differential equation which is at the same time unsolved for the derivative. This circumstance obviously generates a nonunique solvability of the period solution problem. This could in most cases be avoided by introducing an additional requirement that the function $v_0(x)$ should take a preassigned value $v_0(0) = v_0^0$ at the point $x = 0$.

15. GENERALIZATIONS OF THE METHOD

In the preceding sections a rather detailed treatment was given to the numerical-analytic method for investigating periodic solutions of ordinary differential equations, second-order equations and countable systems. The possibilities were pointed out of using the method to investigate rotary and rotary-oscillatory motions as well as its extensions to evolutionary systems, systems of integro-differential equations and partial differential equations.

Our presentation would be incomplete without mentioning the results obtained by Martynyuk [105] and Martynyuk, Perestyuk and Kharabovskaya [109] and the studies made by Perestyuk [135], Kurpel [93] and Grechko [48] under the influence of the numerical-analytic method. We think it useful to dwell, if only briefly, on the latter results. Thus Perestyuk [135] has proposed a scheme for constructing periodic solutions of system of equations of the form

$$\frac{dx}{dt} = Ax + P(t, x, y) \tag{15.1}$$

$$\frac{dy}{dt} = Q(t, x, y)$$

where A is a constant matrix the real parts of whose eigenvalues are nonzero.

The scheme of the numerical-analytic method (4.1) for such systems was modified as follows:

$$x_{m+1}(t, x_0, y_0) = x_0 + \int_0^t e^{A(t-s)}[P(s, x_m(s), y_m(s) - \overline{P(s, x_m(s), y_m(s))}]\, ds$$

$$y_{m+1}(t, x_0, y_0) = y_0 + \int_0^t [Q(s, x_m(s), y_m(s)) - \overline{Q(s, x_m(s), y_m(s))}]\, ds \tag{15.2}$$

where

$$\overline{P(t)} = \frac{A}{e^{AT} - E} \int_0^T e^{A(T-s)} P(s)\, ds, \quad \overline{Q(t)} = \frac{1}{T} \int_0^T Q(s)\, ds \tag{15.3}$$

with E a unit matrix.

Periodic solutions of system (2.4) are determined by points (x_0, y_0) which are roots of the equations

$$Ax_0 + \overline{P(t, x(t), y(t))} = 0$$

$$\overline{Q(t, x(t), y(t))} = 0 \tag{15.4}$$

It should be noted that in a number of cases such a modification of the process extends the conditions of using the numerical-analytic method.

N. S. Kurpel [93] has modified the iteration process (4.1) so that an approximation to a periodic solution should be two-sided, i. e. embracing the solution being sought coordinate-wise from below and above. This is achieved by representing the right-hand side of a system of equations (2.4) in the form of the function $f(t, x, x = y)$ which is non-decreasing in x and non-increasing in y.

Construction of approximations proceeds according to the formulas

$$u_{m+1}(t) = x_0 + \left(1 - \frac{t - t_0}{T}\right) \int_{t_0}^{t} f(s, u_m(s), v_m(s))\, ds$$

$$- \frac{t - t_0}{T} \int_{t}^{t_0 + T} f(s, v_m(s), u_m(s))\, ds$$

$$v_{m+1}(t) = x_0 + \left(1 - \frac{t - t_0}{T}\right) \int_{t_0}^{t} f(s, v_m(s), u_m(s))\, ds$$

$$- \frac{t - t_0}{T} \int_{t}^{t_0 + T} f(s, u_m(s), v_m(s))\, ds \tag{15.5}$$

where well-established functions are taken as initial approximations $u_0(t)$, $v_0(t)$.

The true periodic solution $x(t, x_0)$ satisfies the inequality

$$u_m(t) \leqslant x(t, x_0) \leqslant v_m(t) \tag{15.6}$$

for all $m = 1, 2, \ldots, t \in [0, T]$.

Grechko [48] has made an attempt to combine the iteration scheme of the numerical-analytic method and the Galerkin projection method [198, 202]. As a result, periodic solutions of the system of equations (2.4) are sought using the scheme

$$x_{m+1}(t, x_0) = x_0 + \int_0^t P_r f(s, x_{m+1}(s, x_0))\, ds$$

$$+ \int_0^t [(I - P_r) f(s, x_m(s, x_0)) - \overline{f(s, x_m(s, x_0))}]\, ds \tag{15.7}$$

$x_0(t, x_0) = x_0$; I is the identity operator; P_r is the operator which assigns to any periodic function $x(t)$ with the Fourier series

$$x(t) = \sum_{k=0}^{\infty} (\alpha_k \cos k\omega t + \beta_k \sin k\omega t) \tag{15.8}$$

the trigonometric sum

$$P_r x = \sum_{k=1}^{r} (\alpha_k \cos k\omega t + \beta_k \sin k\omega t) \tag{15.9}$$

Choosing r allows one to accelerate the convergence of the process of approximations to the periodic solution.

II

COLLOCATION METHOD FOR NONLINEAR PERIODIC SYSTEMS

16. INTRODUCTORY REMARKS

The method presented in Chapter I is an iteration one. The collocation method to be described in the present chapter belongs to projection ones. It was first developed in 1934 by Kantorovich [59] to construct solutions of boundary value problems of linear differential equations and three years later again proposed for the same purpose by the American mathematicians R.A. Frazer, W.P. Jones and S.W. Skan [41].

The method as applied to boundary value problems for ordinary differential equations consists essentially in seeking the solution of the problem in the form of a linear combination of basic functions whose coefficients are found from the condition that the solution should exactly satisfy the differential equations at certain points referred to as nodes of collocation. It follows from the above that it can be interpreted as the Galerkin method [84, 205, 206] and, therefore, regarded as numerical-analytic method.

We shall distinguish between the method of algebraic collocation when the basic functions are polynomials, and the method of trigonometric collocation when the basic functions are trigonometric polynomials.

The questions of convergence of the algebraic collocation method for linear differential, integral and integro-differential equations were considered by Karpilovskaya [63, 64], Kis [72, 73] and Vainikko [205]. Different variants and modifications of the collocation method for constructing solutions of linear differential, integral, integro-differential and operator equations have been proposed and studied by Petersen [137], Karpilovskaya [65, 66], Kaspshitskaya [67], Kaspshitskaya and Tukalevskaya [69], Kaspshitskaya and Luchka [68], Shindler [171, 172], Yartsev [219, 220], Elliot and Warne [35], Kadner [57] and Fanta and Kis [36]. The convergence of the

method for multidimensional integral equations in a domain of arbitrary form has been established by Vainikko [207].

The algebraic collocation method for solving nonlinear differential equations was used for the first time and substantiated by Vainikko [206].

In linear and nonlinear boundary value problems Rusell and Shampine [157] have proved the convergence of the collocation method which permits determination of approximate solutions in the form of spline functions [1]. For particular cases these results have been established by De Boor [32] and Bellman and Kalaba [8]. Miroshnichenko [117, 118] has applied the method of collocation and approximation in the form of splines to linear differential equations with retarded argument.

It should be noted that foundation of the algebraic collocation method can be elicited from the theory of projection methods credit for the creation and development of which must go to Polsky [141, 142], Krasnoselsky [79, 84, 85], Mikhlin [115, 116], Kantorovich [60], Vainikko [203, 204, 208] and their followers [30, 54, 74, 94, 98, 114, 168, 224].

Constructing periodic solutions by the collocation method naturally leads to the method of trigonometric collocation.

Considering the important role periodic solutions play in the theory of nonlinear oscillations one could expect to find comprehensive studies dealing with substantiation of this method and its applications. As to particular numerical evaluations, in theoretical electrical engineering works by Pukhov [145, 146], Borkovsky [21, 22] Bondarenko [19, 20] and other authors [44, 177, 178] make wide use of a method of calculating steady-state periodic conditions, known as a point method, closely connected with the method of trigonometric collocation. We are also familiar with a paper by Wohlhart [217] which attempts to find periodic solutions of second-order weakly nonlinear differential equations using the trigonometric collocation method. These works are not concerned, however, with questions of rigorously mathematical foundation. This led us to investigate the applicability and foundation of the trigonometric collocation method for finding periodic solutions of systems of nonlinear differential equations of the general form [150-155].

In what follows we proceed to present the results of those studies.

17. SOME STATEMENTS FROM THE THEORY OF APPROXIMATE SOLUTIONS

Let K be a nonlinear completely continuous operator which acts from a Banach space B with a norm $\|\cdot\|$ into another Banach space B', and L and L_n be linear continuous operators mapping the space B' into B. Also let the superposition operators

$$T = LK, \qquad T_n = L_n K \tag{17.1}$$

act in the space B.

Consider a pair of equations

$$v = Tv \tag{17.2}$$

$$v = T_n v \tag{17.3}$$

Theorem 17.1 [206]. *Let equation* (17.2) *have in* B *a solution* $v = v^0$ *with a nonzero index and unique in the ball*

$$\| v - v^0 \| \leqslant \sigma \tag{17.4}$$

Suppose that the sequence of operators L_n *strongly converges to the operator* L

$$\| L_n - L \| \underset{n \to \infty}{\to} 0 \tag{17.5}$$

Then

1. *With* n *sufficiently large* $(n \geqslant n_0)$ *equation* (17.3) *has a solution* $v_n = v_n^0$ *which satisfies the inequality*

$$\| v_n^0 - v^0 \| \leqslant \sigma \tag{17.6}$$

and the relation

$$\| v_n^0 - v^0 \| \underset{n \to \infty}{\to} 0 \tag{17.7}$$

2. *If the operator* K *is differentiable (according to Fréchet) at the point* v^0 *and the homogeneous equation*

$$v - LK'(v^0)v = 0 \tag{17.8}$$

has only a trivial solution, then the speed of convergence is estimated from the inequality

$$\| v_n^0 - v^0 \| \leqslant c \| (L_n - L) K v^0 \| \tag{17.9}$$

$$c = \text{const}, \qquad n = n_0, \qquad n_0 + 1, \ldots$$

3. *If the operator K is continuously differentiable at the point v^0, then, with n sufficiently large, the solution v_n^0 of equation* (17.3) *is unique in the ball* (17.4).

Let Ω be a nonempty open set of the Banach space B and K a nonlinear operator which acts in B and is continuous in Ω.

Let B_n be a sequence of closed subspaces B, $\Omega_n = \Omega \bigcap B_n$, K_n an operator acting in B_n and continuous in Ω_n, P_n a linear operator projecting the space B onto the subspace B_n, and S_n and U_n operators of the form

$$S_n = K_n - P_n K \tag{17.10}$$

$$U_n = K - P_n K \tag{17.11}$$

which act in the spaces B_n and B, respectively.

Consider the equations

$$v = Kv \tag{17.12}$$

$$v = K_n v \tag{17.13}$$

Theorem 17.2 [85]. *Let $v = v_n \in B_n$ be an exact or approximate solution of equation* (17.13). *Suppose that*

1. *The operators K, $P_n K$ and K_n are differentiable (according to Frechet) in some neighbourhood of the point $v_n \in \Omega_n$.*

2. *The operator $I - K_n'(v_n)$ is continuously convertible in B_n*

$$\gamma_n \equiv (1 + \varkappa_n \| P_n K'(v_n) \|) \| U_n'(v_n) \| + \varkappa_n \| S_n'(v_n) \| < 1 \tag{17.14}$$

3. *Some $\delta_n > 0$, $0 \leqslant q_n < 1$ satisfy the inequalities*

$$\sup_{\| v - v_n \| \leqslant \delta_n} \| K'(v) - K'(v_n) \| \leqslant \frac{q_n}{\varkappa_n'} \tag{17.15}$$

$$\| v_n - K v_n \| \leqslant \frac{\delta_n (1 - q_n)}{\varkappa_n'} \tag{17.16}$$

where

$$\varkappa_n = \| [I - K_n'(v_n)]^{-1} \| \tag{17.17}$$

and I is the identity operator;

$$\varkappa_n' = \frac{1 + \varkappa_n \| P_n K'(v_n) \|}{1 - \gamma_n} \tag{17.18}$$

Then in the ball $\|v - v_n\| \leqslant \delta_n$ *equation* (17.12) *has a unique solution* $v = v^0$ *for which the following estimate is valid*

$$\frac{\alpha_n}{1 + q_n} \leqslant \|v_n - v^0\| \leqslant \frac{\alpha_n}{1 - q_n} \tag{17.19}$$

where

$$\alpha_n = \|(I - K'(v_n))^{-1}(v_n - Kv_n)\| \leqslant \varkappa_n' \|v_n - Kv_n\| \tag{17.20}$$

The theorems will be used below to substantiate the trigonometric collocation method as applied to the problem of constructing periodic solutions of systems of nonlinear differential equations.

Let $C[a, b]$ be a Banach space of all functions $f(t)$ continuous in $[a, b]$ with the metric

$$|f(t)|_0 = \max_{t \in [a, b]} |f(t)| \tag{17.21}$$

We denote by $H(P_m)$ a set of all polynomials of degree not higher than m and by $E_m(f, P_m)$ the deviation of the function $f(t) \in C[a, b]$ from the polynomial $P_m(t) \in H(P_m)$

$$E_m(f, P_m) = |f(t) - P_m(t)|_0 \tag{17.22}$$

It will be recalled [9] that there exists a unique polynomial $P_m^0(t) \in H(P_m)$ for which

$$|f(t) - P_m^0(t)|_0 = \inf_{P_m(t) \in H(P_m)} E_m(f, P_m) = E_m(f) \tag{17.23}$$

This polynomial is said to be a polynomial of the best uniform approximation to the function $f(t)$ in the polynomial class of degree not higher than m, and the number $E_m(f)$ is said to be the best approximation.

Similarly, there exists a unique trigonometric polynomial of the best uniform approximation among all trigonometric polynomials (of period T) of degree not higher than m for a continuous periodic function $f(t)$ of period T. It is well-known that $E_m(f) \underset{m \to \infty}{\to} 0$ and that the speed of this tendency is essentially dependent on the smoothness of the function. Thus, according to the Jackson theorem [46],

$$E_m(f) = 0\left(\frac{1}{m^{k+\alpha}}\right) \tag{17.24}$$

if the function $f(t)$ belongs to the space of functions k times continuously differentiable whose kth derivatives satisfy a Hölder-Lipschitz condition with the exponent $\alpha (0 < \alpha \leqslant 1)$.

The best approximations will be used below when estimating deviations of exact solutions from their approximate values obtained by the collocation method.

18. A TWO-POINT BOUNDARY VALUE PROBLEM FOR A SYSTEM OF NONLINEAR DIFFERENTIAL EQUATIONS

The problem of periodic solutions of systems of nonlinear differential equations with right-hand sides periodic in t is a special case of the two-point boundary value problem. The latter was solved by the algebraic collocation method for an mth-order nonlinear differential equation by Vainikko [206]. In this section the results of [206] are extended to include systems of nonlinear differential equations.

Suppose it is necessary to find a solution of the system of equations

$$\frac{dx}{dt} = f(t, x) \tag{18.1}$$

which satisfies the boundary conditions

$$Ax(a) + Bx(b) = 0 \tag{18.2}$$

where $x = (x_1, x_2, \ldots, x_n) \in E_n$, $f(t, x) = (f_1(t, x), \ldots, f_n(t, x))$ is a vector function with values in E_n, defined and continuous in t, x for $t \in [a, b]$, $x \in D \subset E_n$; $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$, $B = \operatorname{diag}(b_1, b_2, \ldots, b_n)$ are diagonal matrices; D is a closed bounded domain of the space E_n.

The solution of the posed problem (18.1), (18.2) is sought approximately in the form of a polynomial

$$x_m(t) = \sum_{k=0}^{m} Q_k \varphi_k(t) \tag{18.3}$$

where $\varphi_k(t) = (\varphi_{k1}(t), \varphi_{k2}(t), \ldots, \varphi_{kn}(t))$, $k = 0, 1, \ldots, m$ are vector polynomials of degree $(k + 1)$ chosen from the condition that they should satisfy the boundary condition (18.2)

$$A\varphi_k(a) + B\varphi_k(b) = 0 \tag{18.4}$$

$$Q_k = \operatorname{diag}(Q_{k1}, Q_{k2}, \ldots, Q_{kn})$$

are constant diagonal matrices to be determined.

Since the polynomials $\varphi_k(t)$, $k = 0, 1, \ldots, m$ satisfy the boundary conditions (18.2) and the matrices A, B, Q_k commute, the function $x_m(t)$ also satisfies the boundary conditions. The polynomials $\varphi_k(t)$ are, obviously, chosen nonuniquely.

One can, indeed, take as $\varphi_k(t)$ $(k \geqslant 1)$ a vector function with the same components equal to

$$(t-a)(t-b)P_{k-1}(t) \tag{18.5}$$

where $P_{k-1}(t)$ is an arbitrary polynomial of degree $(k-1)$. The function

$$\varphi_0(t) = (\alpha_1 + \beta_1 t, \ldots, \alpha_n + \beta_n t) \tag{18.6}$$

which satisfies the boundary condition does not always exist. For it to exist it is necessary and sufficient that one of the following inequalities should hold

$$\det(A+B) \neq 0 \tag{18.7}$$

$$\det(aA + bB) \neq 0 \tag{18.8}$$

When one of these inequalities holds, $\varphi_0(t)$ is determined to within an arbitrary constant.

According to the general scheme of the collocation method, unknown coefficients Q_k are determined from the condition that the discrepancy

$$\frac{dx_m(t)}{dt} - f(t, x_m(t)) \tag{18.9}$$

should vanish at the fixed $(m+1)$ points of the collocation

$$t_0, t_1, \ldots, t_m \tag{18.10}$$

i. e. from the system of nonlinear equations

$$\frac{dx_m(t_i)}{dt} = f(t, x_m(t_i)), \quad i = 0, 1, 2, \ldots, m \tag{18.11}$$

The solvability of this system of equations and the convergence of approximations $x_m(t)$ to an exact solution of the boundary value problem with $m \to \infty$ depends in many respects on the choice of collocation nodes. In the algebraic collocation method considered here this can be accomplished by choosing as nodes (18.10) the roots of the polynomial $P_{m+1}(t, \rho)$ of degree $(m+1)$ which belongs to the system of orthogonal polynomials of weight $\rho(t)$ in the interval $[a, b]$. It is well-known [9] that if $\rho(t)$ is a nonnegative

summable function which satisfies the condition

$$\int_a^b \frac{ds}{\rho(s)} < \infty \tag{18.12}$$

then the polynomial $P_{m+1}(t, \rho)$ has exactly $(m+1)$ different roots in the interval $[a, b]$.

Let collocation nodes (18.10) be chosen according to a system of orthogonal functions with a weight $\rho(t)$ in $[a, b]$. Suppose, in addition, that the boundary value problem (18.1), (18.2) has a solution $x^0(t)$ which belongs to the domain D together with its neighbourhood. We seek the conditions under which the system of equations (18.11) is solvable and estimate the deviation of $x_m(t)$ from $x^0(t)$.

As before, we denote by $C[a, b]$ the space of vector functions

$$x(t) = (x_1(t), \ldots, x_n(t)) \tag{18.13}$$

continuous in the interval $[a, b]$ with the norm

$$|x(t)|_C = \max_{i=1,2,\ldots,n} \max_{t\in[a,b]} |x_i(t)| = \max_{i=1,2,\ldots,n} |x_i(t)|_0 \tag{18.14}$$

and by $|x(t)|_0$ we mean the vector

$$|x(t)|_0 = \left(\max_{t\in[a,b]} |x_1(t)|, \ldots, \max_{t\in[a,b]} |x_n(t)|\right) \tag{18.15}$$

We shall also need the space $L_\rho^2[a, b]$ of quadratically summable vector functions with a weight $\rho(t)$ in the interval $[a, b]$, for which by $|x(t)|_2$ we shall denote the norm

$$|x(t)|_2 = \max_{i=1,2,\ldots,n} \left[\int_a^b |\rho(s)x_i(s)|^2 ds\right]^{1/2} \tag{18.16}$$

As before, inequalities between vectors will be understood componentwise. For the vector function (18.13) the best uniform approximation $E_m(x)$ is given by the vector

$$E_m(x) = (E_m(x_1), \ldots, E_m(x_n))$$

where $E_m(x_k)$ is the best uniform approximation to the function $x_k(t)$ and

$$E_m^0(x) = \max_{k=1,2,\ldots,n} E_m(x_k) \tag{18.17}$$

Theorem 18.1. *Suppose that*

1. *The boundary value problem* (18.1), (18.2) *has the solution* $x^0 = x^0(t)$.

2. *The function* $f(t, x)$ *and the matrix* $(\partial f(t, x)/\partial x) = F(t, x)$ are *defined and continuous when*

$$t \in [a, b], \qquad |x - x^0|_C \leqslant \delta, \quad \delta > 0 \tag{18.18}$$

3. *The system of equations in variations with respect to the solution* $x^0(t)$

$$\frac{dx}{dt} = F(t, x^0)x \tag{18.19}$$

has only a zero solution under boundary conditions (18.2).

4. *The matrices* A, B *are such that* $\det(A + B) \neq 0$.

Then

(1) *one can indicate* $\varkappa > 0$ *such that in the ball*

$$\left| \frac{dx}{dt} - \frac{dx^0}{dt} \right|_2 \leqslant \varkappa \tag{18.20}$$

of the functions $x(t)$ *absolutely continuous in* $[a, b]$ *the solution* $x^0(t)$ *of the boundary value problem is unique;*

(2) *with* m *sufficiently large* $(m \geqslant m_0)$ *the collocation method* (18.11), (18.3) *determines the unique approximate solution* $x_m = x_m(t)$ *in* (18.20);

(3) *the sequence of approximations* $x_m(t)$ *converges uniformly and* $\frac{dx_m(t)}{dt}$ *converges in the metric* $L_p^2[a, b]$ *with* $m \to \infty$ *to* $x^0(t)$ *and* $\frac{dx^0(t)}{dt}$, *respectively, with*

$$|x_m(t) - x^0(t)|_C \leqslant c_1 E_m^0\left(\frac{dx^0}{dt}\right) \tag{18.21}$$

$$\left| \frac{dx_m(t)}{dt} - \frac{dx^0(t)}{dt} \right|_2 \leqslant c_2 E_m^0\left(\frac{dx^0}{dt}\right) \tag{18.22}$$

where c_1, c_2 *are* m*-independent constants.*

The proof of the theorem is based on the reduction of the initial boundary value problem and the system of equations (18.11) to two operator equations, equivalent to them, in the space $L_p^2[a, b]$, followed by the application of Theorem 17.1. To do this we

introduce into analysis the Green function $G(t, s)$ of the boundary value problem (18.2) for the system of equations

$$\frac{dx}{dt} = 0 \tag{18.23}$$

This function has the form

$$G(t, s) = \begin{cases} E - (A + B)^{-1}B \text{ when } s \leqslant t \\ \quad -(A + B)^{-1}B \text{ when } s \geqslant t \end{cases} \tag{18.24}$$

where E is a unit matrix.

It follows from the form of the Green function $G(t, s)$ that the operator G

$$Gv = \int_a^b G(t, s)\, v(s)\, ds \tag{18.25}$$

is a linear completely continuous operator from the space $L_p^2[a, b]$ into the space $C[a, b]$.

Put $v^0(t) = \dfrac{dx^0(t)}{dt}$; then

$$x^0 = Gv^0 \tag{18.26}$$

In view of the boundedness of the operator G we can choose in the space $L_p^2[a, b]$ the ball

$$|v - v^0|_2 \leqslant \varkappa \tag{18.27}$$

of a radius $\varkappa$ so small that the functions $x = Gv$ should satisfy condition (18.18).

Assign in the ball (18.27) of the space $L_p^2[a, b]$ the operator K which translates the functions $v(t)$ from this ball into those of the space $C[a, b]$ by the formula

$$Kv = f\left(t, \int_a^b G(t, s)\, v(s)\, ds\right) \tag{18.28}$$

It follows from the vector function $f(t, x)$ being continuous and the operator G completely continuous that the operator K is completely continuous in the domain (18.27).

If the operator of embedding the space $C[a, b]$ into the space $L_p^2[a, b]$ is denoted by P, then the problem (18.1), (18.2) is

equivalent to the operator equation

$$v = PKv \tag{18.29}$$

solved in the space $L_\rho^2[a, b]$.

The operator P, being an embedding operator, is linear and continuous. The solutions of the initial boundary value problem and that of the operator equation (18.29) are, obviously, connected by relation (18.26).

On putting $v_m(t) = dx_m(t)/dt$, we obtain

$$x_m = Gv_m \tag{18.30}$$

The system of algebraic equations of the collocation method (18.11) can now be re-written in the form

$$P_m v_m = P_m K v_m \tag{18.31}$$

where P_m is linear operator which juxtaposes each function continuous in $[a, b]$ its interpolation Lagrange polynomial of degree not higher than m constructed using the nodes under consideration, namely the roots of the orthogonal polynomial $P_{m+1}(t, \rho)$.

In our case, by the Erdös-Turán theorem [130], the interpolation Lagrange polynomial converges in a root-mean-square fashion with weight $\rho(t)$ to the function being approximated. This is sufficient for the sequence of the operators $P_m : C[a, b] \to L_\rho^2[a, b]$ to tend strongly to the operator P and hence for the inequality

$$\| P_m \| \leqslant c_4 = \text{const}, \quad m = 0, 1, 2, \ldots \tag{18.32}$$

to occur by the Banach-Steinhaus theorem [61].

Since $x_m(t)$ is a polynomial of degree $(m + 1)$, $v_m(t) = dx_m(t)/dt$ is a polynomial of degree not higher than m; hence

$$P_m v_m = v_m \tag{18.33}$$

which makes it possible to write the operator equation (18.31) in the form

$$v_m = P_m K v_m \tag{18.34}$$

The assumption that the system of equations (18.19) has, under the boundary conditions (18.2), only a trivial solution leads to the fact that similarly the equation

$$v - PK'(v^0)v = 0 \tag{18.35}$$

where

$$K'(v) = \left(\frac{\partial f(t, Gv)}{\partial x}\right) G$$

has only a trivial solution, too.

The continuity of Jacobi matrix $F(t, x)$ in the domain (18.18) implies the continuous differentiability of the operator K in the ball $|v - v^0|_2 \leqslant \varkappa$. This means that the operator equations (18.29) and (18.34) satisfy all the conditions of Theorem 17.1. (The isolation of the solution v^0 follows from the fact that equation (18.35) has only a zero solution [81]). According to the statements of this theorem: (1) the solution $v = v^0$ of equation (18.29) is unique in the ball (18.27); (2) with $m \geqslant m_0$ sufficiently large, equation (18.34) has also a unique solution v_m in the ball $|v - v^0|_2 \leqslant \varkappa$, the solution tending to v^0 when $m \to \infty$ by the norm of the space $L_\rho^2[a, b]$

$$|v_m - v^0|_2 \underset{m \to \infty}{\to} 0 \tag{18.36}$$

(3) the following estimate is valid for the deviation of v_m from v^0

$$|v_m - v^0|_2 \leqslant c_3 |v^0 - P_m P^{-1} v^0|_2, \quad m \geqslant m_0 \tag{18.37}$$

Let $p_m(t) = (p_{m1}(t), p_{m2}(t), \ldots, p_{mn}(t))$ be an arbitrary vector polynomial of degree not higher than m. Then $P_m P^{-1} p_m(t) = p_m(t)$; and relation (18.22) we are proving follows from inequality (18.37)

$$\begin{aligned}
|v_m - v^0|_2 &\leqslant c_3 (|v^0(t) - p_m(t)|_2 + |P_m P^{-1}(v^0 - p_m(t))|_2) \\
&= c_3 \max_{i=1,2,\ldots,n} (|v_i^0(t) - p_{mi}(t)|_2 \\
&\quad + |P_m P^{-1}(v_i^0(t - p_{mi}(t))|_2) \\
&\leqslant c_3 \max_{i=1,2,\ldots,n} |v_i^0(t) - p_{mi}(t)|_0 \left[\left(\int_a^b \rho(s)\,ds\right)^{1/2} + c_4\right] \\
&\leqslant c_3 \left[\left(\int_0^b \rho(s)\,ds\right)^{1/2} + C_4\right] \max_{i=1,2,\ldots,n} E_m(v_i^0) \\
&\leqslant c_2 E_m^0(v^0)
\end{aligned}$$

To obtain estimate (18.21), we take into account the relation

$$x_m(t) - x_0(t) = \int_a^b G(t, s)(v_m(s) - v^0(s))\, ds \tag{18.38}$$

owing to which

$$|x_m(t) - x^0(t)|_C$$

$$= \max_{i=1,2,\ldots,n} |x_{mi}(t) - x_i^0(t)|_0$$

$$= \max_{i=1,2,\ldots,n} \left| \sum_{j=1}^{n} \int_a^b G_{ij}(t, s)(v_{mj}(s) - v_j^0(s))\, ds \right|_0$$

$$= \max_{i=1,2,\ldots,n} \left| \sum_{j=1}^{n} \int_a^b \frac{G_{ij}(t, s)}{[\rho(s)]^{1/2}} [\rho(s)]^{1/2} (v_{mj}(s) - v_j^0(s))\, ds \right|_0 \tag{18.39}$$

where $G_{ij}(t, s)$ are elements of the Green matrix $G(t, s)$.

From (18.39), by the Cauchy-Buniakowski inequality, we obtain

$$|x_m(t) - x^0(t)|_C$$

$$\leqslant \max_{i=1,2,\ldots,n} \left| \sum_{j=1}^{n} \left(\int_a^b \frac{|G_{ij}(t, s)|^2}{\rho(s)} ds \right)^{1/2} \right.$$

$$\left. \times |v_{mj} - v_j^0|_2 \right|_0$$

Put

$$\max_{j=1,2,\ldots,n} \left| \left(\int_a^b \frac{|G_{ij}(t, s)|^2}{\rho(s)} ds \right)^{1/2} \right|_0 = c_i$$

then

$$|x_m(t) - x^0(t)|_C \leqslant \max_{i=1,2,\ldots,n} c_i \left(\sum_{j=1}^{n} |v_{mj} - v_j^0|_2 \right)$$

and hence

$$|x_m(t) - x^0(t)|_C = \max_{i=1,2,\ldots,n} c_i (m c_2 E_m^0(v^0)) = c_1 E_m^0(v^0)$$

which completes the proof of the theorem.

We make some remarks concerning the boundary value problem under consideration.

Remark 18.1. The matrices A and B contained in the boundary conditions (18.2) were supposed to be diagonal. They may be any square matrices which satisfy the condition

$$\det(A + B) \neq 0$$

In the latter case an approximate solution may be sought as a vector polynomial of degree m of the form

$$x_m(t) = (A + B)^{-1} \sum_{k=1}^{m} (-a^k A - b^k B) Q_k + \sum_{k=1}^{m} Q_k t^k \qquad (18.40)$$

It is evident that the vector polynomial $x_m(t)$ so taken satisfies the boundary conditions (18.2) with arbitrary diagonal matrices Q_k.

Remark 18.2. The proof of Theorem 18.1 was essentially based on the fact that the degree of the polynomial – of the approximate solution – does not exceed the number of unknown coefficients. Had this correspondcnce been violated, had we, for instance, sought the approximate solution in the form

$$x_m(t) = (t - a)(t - b)(Q_0 + Q_1 t + \ldots + Q_m t^m)$$

the proof of the theorem would have failed since relation (18.33) would not have been valid.

Remark 18.3. Theorem 18.1. can be extended to include the multipoint boundary value problem when the boundary conditions are given in the form

$$\sum_{i=0}^{\nu} A_i x(\tau_i) = d, \quad a = \tau_0 < \tau_1 < \ldots < \tau_\nu = b \qquad (18.41)$$

In this case it is naturally supposed that the square matrices A_i satisfy the condition

$$\det\left(R = \sum_{i=0}^{\nu} A_i\right) \neq 0 \qquad (18.42)$$

The approximate solution of the multipoint problem is sought in the form

$$x_m(t) = d + R^{-1} \sum_{k=1}^{m} H_k Q_k + \sum_{k=1}^{m} Q_k t^k \qquad (18.43)$$

where

$$H_k = -\sum_{i=0}^{\nu} \tau_i^k A_i$$

In proving the theorem similar to Theorem 18.1 for the multipoint boundary value problem one should take into account the fact that the Green function $G(t, s)$ now has the form

$$G(t, s) = \begin{cases} E - R^{-1} \sum_{i=k}^{\nu} A_i \text{ when } s \leqslant t, \tau_{k-1} \leqslant s < \tau_k \\ -R^{-1} \sum_{i=k}^{\nu} A_i \text{ when } s \geqslant t, \tau_{k-1} \leqslant s < \tau_k \end{cases}$$

and that the operator G is given by the equality

$$Gv = R^{-1}d + \int_a^b G(t, s)v(s)\,ds$$

The rest of the reasoning is essentially unchanged.

19. CONSTRUCTION OF PERIODIC SOLUTIONS USING THE METHOD OF TRIGONOMETRIC COLLOCATION

Suppose that the system of equations (18.1) has the right-hand side periodic in t of period T. The boundary value problem solved in the previous section supposes that $\det(A + B) \neq 0$. The problem of periodic solutions of period T of the system of equations (18.1) leads to the boundary conditions

$$x(0) - x(T) = 0 \tag{19.1}$$

consequently, it is required that it should be considered when the matrix $A + B \equiv 0$.

So, given the system of differential equations (18.1) the right-hand side of which is continuous and periodic in t of period T. We shall suppose that such a solution, $x = x^0(t)$, exists. We now present the trigonometric collocation method for its approximate construction.

According to the general scheme of the collocation method an approximation to the solution $x^0(t)$ is sought in the form

$$x_m(t) = a_0 + \sum_{j=1}^{m} (a_j \cos j\omega t + b_j \sin j\omega t) \tag{19.2}$$

where

$$\omega = \frac{2\pi}{T}, \quad a_j = (a_{j1}, a_{j2}, \ldots, a_{jn})$$

$$b_j = (b_{j1}, b_{j2}, \ldots, b_{jn}), \quad x_m(t) = (x_{m1}(t), \ldots, x_{mn}(t))$$

Denote by x_j^Γ the vector of coefficients, ordering them as follows:

$$x_j^\Gamma = (a_{0j}, a_{1j}, b_{1j}, \ldots, a_{mj}, b_{mj}) \tag{19.3}$$

$$j = 1, 2, \ldots, n$$

The coefficients are determined from the condition that the discrepancy should vanish at the collocation nodes $t_0, t_1, \ldots, t_{N-1}$ chosen equidistant in the interval of the period

$$t_i = i\frac{T}{N}, \quad i = 0, 1, \ldots, N-1, \quad N = 2m+1 \tag{19.4}$$

Hence the coefficients are determined from the system of equations

$$\frac{dx_m(t_i)}{dt} = f(t_i, x_m(t_i)), \quad i = 0, 1, \ldots, N-1 \tag{19.5}$$

The choice of equidistant points as the collocation nodes ensures that the method of trigonometric collocation converges. A theorem similar to Theorem 18.1 does in fact hold.

Theorem 19.1. *Suppose that*

1. *The system of differential equations* (18.1) *with the right-hand side periodic in t of period T has a periodic solution of period T, that is,* $x^0 = x^0(t)$.

2. *The function* $f(t, x)$ *and the matrix* $(\partial f(t, x)/\partial x) = F(t, x)$ *are defined and continuous when*

$$t \in [0, T], \qquad |x - x^0|_C \leqslant \delta, \quad \delta > 0 \tag{19.6}$$

$$(|x(t)|_C = \max_{i=1,2,\ldots,n} \max_{t\in[0,T]} |x_i(t)|)$$

3. *All the multipliers of a linear equation in variations with respect to the solution* $x^0(t)$

$$\frac{dx}{dt} = F(t, x^0)x \tag{19.7}$$

are different from one.

Then

(1) *One can indicate* $\varkappa > 0$ *such that in the ball*

$$\left|\frac{dx}{dt} + x - \left(\frac{dx^0}{dt} + x^0\right)\right|_2 \leqslant \varkappa \tag{19.8}$$

$$|x(t)|_2 = \max_{i=1,2,\ldots,n}\left[\int_0^T |x_i(s)|^2\, ds\right]^{1/2}$$

the solution $x^0(t)$ *of the periodic system* (18.1) *is unique.*

(2) *With m sufficiently large* $(m \geqslant m_0)$ *the trigonometric collocation method* (19.5), (19.2), (19.4) *determines a unique approximate periodic solution* $x = x_m(t)$ *in the domain* (19.8).

(3) *The sequence of approximations* $x_m(t)$ *converges uniformly and* $dx_m(t)/dt$ *converges in the metric of the space* $L^2[0, T]$ *when* $m \to \infty$ *to* $x^0(t)$ *and* $dx^0(t)/dt$, *respectively, the following estimates being valid*

$$|x_m(t) - x^0(t)|_C \leqslant c_5 E_m^0(v^0) \tag{19.9}$$

$$\left|\frac{dx_m(t)}{dt} - \frac{dx^0(t)}{dt}\right|_2 \leqslant c_6 E_m^0(v^0) \tag{19.10}$$

where c_5 and c_6 are constants, $v^0(t) = x^0(t) + \dfrac{dx^0(t)}{dt}$.

Proof. Denote by $\tilde{G}(t, s)$ the Green function of the boundary value problem (19.1) for the system of differential equations

$$\frac{dx}{dt} + x = 0 \tag{19.11}$$

It is easy to see that

$$\tilde{G}(t, s) = \begin{cases} e^{-t}\left[1 + (1 - e^{-T})^{-1} e^{-T}\right] e^s E \text{ when } s \leqslant t \\ e^{-t}\left[(1 - e^{-T})^{-1} e^{-T}\right] e^s E \text{ when } s \geqslant t \end{cases} \tag{19.12}$$

By making the substitutions $dx(t)/dt + x(t) = v(t)$ and $dx_m(t)/dt + x_m(t) = v_m(t)$ in the systems of equations (18.1) and (19.5), the given periodic boundary value problem (18.1), (19.1) and the algebraic equations of the trigonometric collocation method (19.5) in the space $L^2[0, T]$ can be reduced to their equivalent operator equations

$$v = RKv \tag{19.13}$$

$$v_m = R_m K v_m \tag{19.14}$$

where $K : L^2[0, T] \to C[0, T]$ is a nonlinear completely continuous operator of the form

$$Kv = f\left(t, \int_0^T \tilde{G}(t, s) v(s)\, ds\right) + \int_0^T \tilde{G}(t, s) v(s) \tag{19.15}$$

R is the linear operator of embedding the space $C[0, T]$ into the space $L^2[0, T]$; and R_m is a linear operator which assigns to each continuous periodic function of period T its trigonometric interpolation polynomial constructed by the equidistant points (19.4).

According to [56], the choice of equidistant points ensures that the trigonometric interpolation polynomial converges in a root-mean-square fashion to the periodic function being approximated. Hence, there is a strong convergence of the sequence of the operators R_m: $C[0, T] \to L^2[0, T]$ to the operator $R : C[0, T] \to L^2[0, T]$. Therefore, by the Banach-Steinhaus theorem, the sequence of the operators R_m is bounded in the totality

$$\|R_m\| \leqslant \text{const}, \qquad m = 0, 1, 2, \ldots \tag{19.16}$$

The further reasoning follows the scheme of the proof of Theorem 18.1 and is omitted here. Thus the theorem is proved.

Remark 19.1. The operator equations (18.29), (18.34), (19.13) and (19.14) were considered in the space L^2. This fact is essentially used in obtaining relations (18.32) and (19.16). Note that if the operator equations are considered in the space C, then relations (18.32) and (19.16) will not be obtained since $P_m : C[a, b] \to C[a, b]$ and $R_m : C[0,T] \to C[0, T]$, being bounded in the norm with each m, will not be bounded in toto.

One point of importance for practical applications of the collocation method should be stressed – that its schemes are easy to extend

to systems of differential equations partially solvable for derivatives

$$\frac{dx}{dt} = f\left(t, x, \frac{dx}{dt}\right) \tag{19.17}$$

If such a system of equations has a periodic solution $x^0 = x^0(t)$, then approximations to this solution can be taken in the same form (19.2) and the coefficients can be determined from the system of equations

$$\frac{dx_m(t_i)}{dt} = f\left(t_i, x_m(t_i), \frac{dx_m(t_i)}{dt}\right) \tag{19.18}$$

The convergence of approximations and the determination of their deviations from $x^0(t)$ are similar to those for the system of differential equations of the form (18.1). This is established by the following theorem which is proved in the same way as Theorem 19.1.

Theorem 19.2. *Suppose that*

1. *The system of differential equations* (19.18) *with a right-hand side periodic in* t *of period* T *has a periodic solution* $x^0 = x^0(t)$ *of period* T.

2. *The function* $f(t, x, y)$ *and the matrices* $(\partial f(t, x, y)/\partial x) = F(t, x, y)$ *and* $(\partial f(t, x, y)/\partial y) = F_1(t, x, y)$ *are defined and continuous when*

$$t \in [0, T], \quad |x - x^0|_C \leqslant \delta, \quad \left|y - \frac{dx^0}{dt}\right| \leqslant \delta \tag{19.19}$$

3. *All the multipliers of the linear equation in variations with respect to the solution* $x^0(t)$

$$\left[E - F_1\left(t, x^0, \frac{dx^0}{dt}\right)\right]\frac{dx}{dt} = F\left(t, x^0, \frac{dx^0}{dt}\right)x \tag{19.20}$$

are different from one. Then

(1) *one can indicate* $\varkappa > 0$ *such that in the ball*

$$\left|\frac{dx}{dt} + x - \left(\frac{dx^0}{dt} + x^0\right)\right|_2 \leqslant \varkappa \tag{19.21}$$

the solution $x^0(t)$ *of the periodic system* (19.17) *is unique;*

(2) *with* m *sufficiently large* $(m \geqslant m_0)$, *the trigonometric collocation method* (19.18), (19.2) *and* (19.4) *determines the unique approximate periodic solution* $x = x_m(t)$ *in the domain* (19.21);

(3) *the sequence of approximations* $x_m(t)$ *converges uniformly and* $dx_m(t)/dt$ *converges in the metric of the space* $L^2[0, T]$ *when* $m \to \infty$ *to* $x^0(t)$ and $dx^0(t)/dt$, *respectively, estimates* (19.9) *and* (19.10) *being valid.*

Note some aspects of applying the trigonometric collocation method to the construction of periodic solutions of special cases of the system of equations (18.1).

If there is one nth order equation with coefficients of period T

$$\frac{d^n x}{dt^n} + c_1(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + c_n(t)x - \psi\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right) = f(t) \tag{19.22}$$

then the approximation to a periodic solution should be taken in the form of the trigonometric polynomial

$$x_m(t) = a_0 + \sum_{j=1}^{m} (a_j \cos j\omega t + b_j \sin j\omega t) \tag{19.23}$$

and the unknown coefficients should be determined from the system of algebraic equations

$$\frac{d^n x_m(t_i)}{dt^n} + \ldots + c_n(t_i)x_m(t_i) - \psi\left(t_i, x_m(t_i), \ldots, \frac{d^{n-1}x_m(t_i)}{dt^{n-1}}\right) = f(t_i) \tag{19.24}$$

$$i = 0, 1, \ldots, 2m$$

In this way we avoid an undesirable increase in the number of functions sought due to going over from equation (19.22) to the system of equations of the form (18.1).

It is easy to obtain for the deviation of the exact periodic solution $x^0(t)$ and its derivatives from their approximations $d^j x_m/dt^j$ $(j = 0, 1, \ldots, n-1)$ the estimates

$$\left| \frac{d^j x_m(t)}{dt^j} - \frac{d^j x^0(t)}{dt^j} \right|_C \leqslant cE_m(v^0) \tag{19.25}$$

where $v^0(t) = d^n x^0(t)/dt + qx^0(t)$; and q is a constant such that the homogeneous equation $d^n x/dt^n + qx = 0$ has only a trivial periodic solution of period T.

Let us now suppose that the periodic system of equations (18.1) is linear

$$\frac{dx}{dt} = A(t)x + f(t) \tag{19.26}$$

Then the system of algebraic equations for determining coefficients of the approximation $x_m(t)$ to the periodic solution is also linear

$$\begin{bmatrix} h_{11}h_{12}\dots h_{1n} \\ h_{21}h_{22}\dots h_{2n} \\ \dots\dots\dots \\ h_{n1}h_{n2}\dots h_{nn} \end{bmatrix} \begin{bmatrix} x_1^\Gamma \\ x_2^\Gamma \\ \dots \\ x_n^\Gamma \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \dots \\ F_n \end{bmatrix} \tag{19.27}$$

Here the following notation is adopted:

$$h_{ij} = \operatorname{diag}[a_{ij}(t_0), \dots, a_{ij}(t_{N-1})]\, M$$

$$h_{ii} = \operatorname{diag}[a_{ii}(t_0), \dots, a_{ii}(t_{N-1})]\, M - M'$$

$$F_j = -(f_j(t_0), f_j(t_1), \dots, f_j(t_{N-1}))$$

$$i, j = 1, 2, \dots, n, \qquad i \neq j \tag{19.28}$$

$$M = \begin{bmatrix} 1 \cos\omega t_0 & \sin\omega t_0 \dots & \cos m\omega t_0 & \sin m\omega t_0 \\ 1 \cos\omega t_1 & \sin\omega t_1 \dots & \cos m\omega t_1 & \sin m\omega t_1 \\ \dots & & & \\ 1 \cos\omega t_{N-1} & \sin\omega t_{N-1} \dots & \cos m\omega t_{N-1} & \sin m\omega t_{N-1} \end{bmatrix}$$

$$M' = \omega \begin{bmatrix} 0 & -\sin\omega t_0 & \cos\omega t_0 \dots & m\sin m\omega t_0 & m\cos m\omega t_0 \\ 0 & -\sin\omega t_1 & \cos\omega t_1 \dots & m\sin m\omega t_1 & m\cos m\omega t_1 \\ \dots & & & & \\ 0 & -\sin\omega t_{N-1} & \cos\omega t_{N-1} \dots & m\sin m\omega t_{N-1} & m\cos m\omega t_{N-1} \end{bmatrix} \tag{19.29}$$

In the case of the linear system (19.26) it is possible to obtain an *a posteriori* error of the approximate periodic solution. Indeed, denote the deviation of the approximate solution $x_m(t)$ from the exact one by

$$\varepsilon_m(t) = x_0(t) - x_m(t) \tag{19.30}$$

Let $\delta_m(t)$ be the discrepancy given by the approximate solution $x_m(t)$ of the system of equations (19.26)

$$\frac{dx_m(t)}{dt} - A(t)\,x_m(t) - f(t) = \delta_m(t) \tag{19.31}$$

Subtracting this equation from the equation

$$\frac{dx_m(t)}{dt} + \frac{d\varepsilon_m(t)}{dt} - A(t)\,x_m(t) - A(t)\,\varepsilon_m(t) - f(t) = 0$$

we obtain for the error $\varepsilon_m(t)$ the equality

$$\frac{d\varepsilon_m(t)}{dt} = A(t)\,\varepsilon_m(t) - \delta_m(t) \tag{19.32}$$

Hence, the *a posteriori* error $\varepsilon_m(t)$ is a periodic solution of the equation

$$\frac{d\varepsilon}{dt} = A(t)\,\varepsilon - \delta_m(t) \tag{19.33}$$

It is not difficult to obtain for this solution the following estimate

$$\|\varepsilon_m(t)\| \leqslant e^{KT}\|C\| + \int_0^t e^{K(t-s)}\|\delta_m\|\,ds$$

$$\leqslant e^{KT}\|C\| + e^{Kt}\frac{1-e^{-Kt}}{K}\|\delta_m\|$$

$$\leqslant e^{KT}\|C\| + \frac{e^{Kt}-1}{K}\|\delta_m\|$$

$$\leqslant K_1\|\delta_m\|$$

where

$$K = \max_{t \in [0, T]} \| A(t) \|, \quad C = [E - \Phi(T)]^{-1} \Phi(T) \int_0^T \Phi^{-1}(s) \delta_m(s) \, ds$$

and $\Phi(t)$ is the fundamental matrix of the homogeneous equation (19.26) such that $\Phi(0) = E$.

20. THE EXISTENCE THEOREM FOR PERIODIC SOLUTIONS

In constructing periodic solutions by the trigonometric collocation method we proceeded from the assumption that such a solution exists. As a result, the system of algebraic equations, which determined the approximation coefficients under sufficiently general conditions, could always be solved and thus made it possible to approach the periodic solution arbitrarily closely.

The question naturally arises as to whether the fact that the system of equations (18.1) has an approximate solution of sufficiently high order found by the trigonometric collocation method implies that there exists an exact periodic solution of the system.

Thorough studies of a similar problem were made by M. Urabe [198, 199], the only difference being that approximate periodic solutions were determined by the Galerkin method. Urabe obtained general theorems on interrelationships between exact periodic solutions and their Galerkin approximations.

We consider the above interrelationships for the trigonometric collocation method. Theorem 19.1 shows that as a rule the existence of an exact periodic solution implies the existence and convergence of approximate solutions obtained by the trigonometric collocation method. We shall prove that the reverse is valid.

Before formulating the last statement we shall specify the necessary notations.

Let $x_m(t)$ be an approximate solution of period T of the periodic system (18.1) obtained on the basis of formulas (19.5), (19.2) and (19.4). For the vector function $f(t, x_m(t))$ denote by $f^m(t, x_m(t))$ a trigonometric vector polynomial of order m whose values at the nodes (19.4) coincide with those of the function $f(t, x_m(t))$.

Put $F_m(t, x_m(t))$ equal to a matrix

$$F_m(t, x_m(t)) = \left(\frac{\partial f^m(t, x_m(t))}{\partial x} \right)$$

and define the operator $\tilde{G}$ which acts on a vector function v as follows:

$$\tilde{G}v = \int_0^T \tilde{G}(t, s)\, v(s)\, ds \tag{20.1}$$

where $\tilde{G}(t, s)$ is the Green matrix (19.12).

Write down the problem of periodic solutions of period T of the system of differential equations (18.1) and the system of algebraic equations of the trigonometric collocation method in the operator form by equations (19.13) and (19.14) solved in the space $L^2[0, T]$.

Theorem 20.1. *Let there exist for a certain m an approximate periodic solution $x_m = x_m(t)$ of period T of the system of differential equations* (18.1) *which has been found by the trigonometric collocation method. Suppose that*

1. *The vector functions $f(t, x)$, $f^m(t, x_m)$, as well as the matrices $F(t, x)$, $F_m(t, x)$, are defined and continuous in a neighbourhood of the function $x_m(t)$*

$$t \in [0, T], \qquad |x - x_m|_C \leqslant \delta_m \tag{20.2}$$

2. *All the multipliers of the linear equation*

$$\frac{dx}{dt} = F_m(t, x_m)\, x \tag{20.3}$$

are different from one.

3. *The following inequality holds*

$$\gamma_m = c(1 + \varkappa_m c\, |F_m(t, x_m) + E|_2)\, |F(t, x_m) - F_m(t, x_m)|_2 < 1 \tag{20.4}$$

4. *For some $\delta_m > 0$, $0 \leqslant q_m < 1$ the inequalities*

$$\sup_{|x - x_m|_C \leqslant \delta_m} |F(t, x) - F(t, x_m)|_C \leqslant \frac{q_m}{\varkappa'_m c} \tag{20.5}$$

$$\left| \frac{dx_m(t)}{dt} - f(t, x_m(t)) \right|_C \leqslant \frac{\delta_m (1 - q_m)}{\varkappa'_m} \tag{20.6}$$

hold where

$$\varkappa'_m = \frac{|1 + \varkappa_m (F_m(t, x_m) + E)\, \tilde{G}|_2}{1 - \gamma_m} \tag{20.7}$$

$$\varkappa_m = |[E - (E + F_m(t, x_m))\,\tilde{G}]^{-1}|_2 \tag{20.8}$$

$$c = |\tilde{G}|_2, \qquad v_m = x_m + \frac{dx_m}{dt} \tag{20.9}$$

Then in the ball

$$|v - v_m|_2 \leqslant \delta_m \tag{20.10}$$

$$\left(v = x + \frac{dx}{dt}\right)$$

the differential equation (18.1) *has a unique periodic solution* $x^0(t)$ *of period* T *such that*

$$|x_m(t) - x^0(t)|_C \leqslant c_7 \frac{\alpha_m}{1 - q_m} \tag{20.11}$$

where c_7 *is an m-independent constant,*

$$\alpha_m \leqslant \varkappa'_m \left| \frac{dx_m}{dt} - f(t, x_m) \right|_2 \tag{20.12}$$

The proof of the theorem consists in verifying the assumptions of Theorem 17.2. We shall show that all of these assumptions hold.

Indeed, $\tilde{G}$ is a bounded operator from $L^2[0, T]$ into the space $C[0, T]$. This allows one to choose the ball (20.10) so that the functions $x = \tilde{G}v$ should satisfy condition (20.2). The operator K

$$Kv = f\left(t, \int_0^T \tilde{G}(t, s)\, v(s)\, ds\right) + \int_0^T \tilde{G}(t, s)\, v(s)\, ds \tag{20.13}$$

is completely continuous on the set of functions of the ball (20.10). It follows from (20.13) that

$$K'(v) = F(t, x)\,\tilde{G} + \tilde{G} \tag{20.14}$$

Assumption 2 of Theorem 20.1 ensures that only a trivial solution of the equation

$$v - R_m K'(v_m)\, v = 0 \tag{20.15}$$

exists in (20.10) and that, therefore, the operator $I - R_m K'(v_m)$ is continuously invertible. In the case at hand the operator S_m of Theorem 17.2 is zero.

Conditions 1-3 of Theorem 17.2 follow from assumptions 1-3 of Theorem 20.1 in an obvious way.

All the conditions of Theorem 17.2 are thus satisfied. The statements of the theorem as applied to our case ensure that all the statements of the theorem being proved are valid. Inequality (20.11) follows from the following estimate

$$\frac{\alpha_r}{1+q_r} \leqslant |v_m - v^0|_2 \leqslant \frac{\alpha_r}{1-q_r} \tag{20.16}$$

on the basis of the relation

$$x_m(t) - x^0(t) = \int_0^T \tilde{G}(t, s)(v_m(s) - v^0(s))\, ds \tag{20.17}$$

Thus the theorem is proved.

21. RELATIONS BETWEEN COEFFICIENTS AND VALUES OF TRIGONOMETRIC POLYNOMIALS

The seeking of approximate periodic solutions of the system of differential equations (18.1) using the trigonometric collocation method involves solving the algebraic system of equations (18.11) set up with respect to the coefficients of the approximate solution $x_m(t)$. Obtaining the system of such equations allows some algorithmization. To begin with, we shall describe the technique of transforming the trigonometric polynomials

$$x_m(t) = a_0 + \sum_{j=1}^{m} (a_j \cos j\omega t + b_j \sin j\omega t) \tag{21.1}$$

which we use below.

Using the coefficients a_j and b_j, we set up the following vector

$$x^{\Gamma} = (a_0,\ a_1,\ b_1, \ldots, a_m,\ b_m) \tag{21.2}$$

and call it the vector of the coefficients of the trigonometric polynomial (21.1). We also introduce into analysis the vector of the values of the polynomial (21.1) at the points (19.4)

$$x^M = (x_m(t_0),\ x_m(t_1), \ldots, x_m(t_{2m})) \tag{21.3}$$

and call it the vector of the values of the trigonometric polynomial (21.1).

The coefficients of the polynomial $x_m(t)$ and its values at the points (19.4) are in one-to-one correspondence [38]. Therefore, there exists a nondegenerate linear transformation realizing the transition from the vector (21.2) to the vector (21.3).

Let

$$x^M = Mx^\Gamma \tag{21.4}$$

$$x^\Gamma = \Gamma x^M \tag{21.5}$$

Considering the expressions for the vectors x^M and x^Γ we see that the elements of the matrices $M = \{M_{pq}\}$ and $\Gamma = \{\Gamma_{pq}\}$ have the form

$$M_{pq} = \begin{cases} 1 & \text{when } q = 1 \\ \cos\left[(p-1)q\dfrac{\pi}{N}\right] & \text{when } q = 2, 4, \ldots, N-1 \\ \sin\left[(p-1)(q-1)\dfrac{\pi}{N}\right] & \text{when } q = 3, 5, \ldots, N \end{cases} \tag{21.6}$$

$$p = 1, 2, \ldots, N, \qquad N = 2m + 1$$

$$\Gamma_{pq} = \begin{cases} \dfrac{1}{N} & \text{when } p = 1 \\ \dfrac{2}{N}\cos\left[p(q-1)\dfrac{\pi}{N}\right] & \text{when } p = 2, 4, \ldots, N-1 \\ \dfrac{2}{N}\sin\left[(p-1)(q-1)\dfrac{\pi}{N}\right] & \text{when } p = 3, 5, \ldots, N \end{cases} \tag{21.7}$$

$$q = 1, 2, \ldots, N$$

We define the three-diagonal matrix l of dimension $N \times N$ by putting

$$l = \begin{bmatrix} \alpha_0 & & & 0 \\ & \alpha_1 & \beta_1 & \\ & -\beta_1 & \alpha_1 & \\ & & \cdot & \\ & & & \cdot \\ & & & \cdot \\ & & & \alpha_m\beta_m \\ 0 & & & -\beta_m\alpha_m \end{bmatrix} \tag{21.8}$$

The vector of coefficients of the derivative $dx_m(t)/dt$ has the form

$$x^{\Gamma}_{(1)} = \omega(0, b_1, -a_1, 2b_2, -2a_2, \ldots, mb_m, -ma_m) \tag{21.9}$$

It is easy to see that the vectors $x^{\Gamma}_{(1)}$ and x^{Γ} are connected by the equality

$$x^{\Gamma}_{(1)} = dx^{\Gamma} \tag{21.10}$$

where d is a matrix of the form (21.8) with $\alpha_0 = 0$, $\alpha_j = 0$, $\beta_j = \omega j$, $j = 1, 2, \ldots, m$.

Let us set up the vector of values $x^{M}_{(1)}$ for the derivative $dx_m(t)/dt$. Using equalities (21.4), (21.10) and (21.5) we have

$$x^{M}_{(1)} = Mx^{\Gamma}_{(1)} = Mdx^{\Gamma} = Md\Gamma x^{M} = Dx^{M} \tag{21.11}$$

where the matrix $D = Md\Gamma$ is a matrix which expresses the vector of values of the derivative $dx_m(t)/dt$ through the vector of values of the function $x_m(t)$.

Equality (21.11) shows that the matrix $\Phi = Md$ allows one to determine the vector of values $x^{M}_{(1)}$ of the derivative $dx_m(t)/dt$ by the vector of coefficients x^{Γ} of the trigonometric polynomial $x_m(t)$.

Similarly, the vector of coefficients $x^{\Gamma}_{(s)}$ of an sth order trigonometric polynomial $x_m(t)$ is expressed through the vector of coefficients x^{Γ} of the function $x_m(t)$

$$x^{\Gamma}_{(s)} = d^s x^{\Gamma} \tag{21.12}$$

where d^s is a matrix of the form (21.8) with

$$\alpha_0 = 0, \quad \beta_j = 0, \quad \alpha_j = (-1)^h j^s \omega^s, \quad j = 1, 2, \ldots, m \tag{21.13}$$

for even $s = 2h$ and

$$\alpha_0 = 0, \quad \alpha_j = 0, \quad \beta_j = (-1)^h j^s \omega^s, \quad j = 1, 2, \ldots, m \tag{21.14}$$

for odd $s = 2h + 1$ $(h = 1, 2, \ldots)$.

The vector of values $x^{M}_{(s)}$ of an sth derivative $d^s x_m(t)/dt^s$ is expressed through the vector of coefficients x^{Γ} of the trigonometric polynomial $x_m(t)$ by the equality

$$x^{M}_{(s)} = Mx^{\Gamma}_{(s)} = Md^s x^{\Gamma} = \Phi^s x^{\Gamma}$$

and through the vector of values x^{M} by the equality

$$x^{M}_{(s)} = Md^s \Gamma x^{M} = D^s x^{M}$$

where

$$\Phi^s = Md^s, \quad D^s = Md^s\Gamma \tag{21.15}$$

We shall give general expressions for the elements of the matrices Φ^s and D^s. We have the following expressions for the product of the matrices $Z = Ml = \{Z_{pk}\}$ and $L = Ml\Gamma = \{L_{pk}\}$

$$Z_{pk} = \begin{cases} \alpha_0 & \text{when } k = 1 \\ \cos\left[(p-1)k\dfrac{\pi}{N}\right]\alpha_{\frac{k}{2}} - \sin\left[(p-1)k\dfrac{\pi}{N}\right]\beta_{\frac{k}{2}} & \text{when } k = 2, 4, \ldots, N-1 \\ \cos\left[(p-1)(k-1)\dfrac{\pi}{N}\right]\beta_{\frac{k-1}{2}} + \sin\left[(p-1)(k-1)\dfrac{\pi}{N}\right]\alpha_{\frac{k-1}{2}} & \text{when } k = 3, 5, \ldots, N \end{cases}$$

$$L_{pk} = \frac{2}{N}\sum_{j=1}^{m}\left\{\alpha_j \cos\left[2j\frac{\pi(p-k)}{N}\right] + \beta_j \sin\left[2j\frac{\pi(k-p)}{N}\right]\right\} + \frac{\alpha_0}{N} \tag{21.16}$$

$$p, k = 1, 2, \ldots, N, \quad N = 2m+1$$

Taking into account the form of the matrix d^s, the elements of the matrices $\Phi^s = \{\Phi^s_{pk}\}$ and $D^s = \{D^s_{pk}\}$ are found.

For even $s (s = 2h)$ we have

$$\Phi^s_{pk} = \begin{cases} 0 & \text{when } k = 1 \\ \cos\left[(p-1)k\dfrac{\pi}{N}\right](-1)^h\left(\dfrac{k}{2}\right)^s\omega^s & \text{when } k = 2, 4, \ldots, N-1 \\ \sin\left[(p-1)(k-1)\dfrac{\pi}{N}\right](-1)^h\left(\dfrac{k-1}{2}\right)^s\omega^s & \text{when } k = 3, 5, \ldots, N \end{cases}$$

for odd $s(s = 2h + 1)$ we have

$$\Phi_{pk}^{s} = \begin{cases} 0 & \text{when } k = 1 \\ \sin\left[(p-1)k\dfrac{\pi}{N}\right](-1)^{h+1}\left(\dfrac{k}{2}\right)^{s}\omega^{s} & \text{when } k = 2,\ 4,\ \ldots,\ N-1 \\ \cos\left[(p-1)(k-1)\dfrac{\pi}{N}\right](-1)^{h}\left(\dfrac{k-1}{2}\right)^{s}\omega^{s} & \text{when } k = 3,\ 5,\ \ldots,\ N \end{cases}$$

$$D_{pk}^{s} = \omega^{s}\begin{cases} \dfrac{2}{N}(-1)^{h}\sum\limits_{j=1}^{m} j^{s}\cos\left[2j\dfrac{\pi(p-k)}{N}\right] & \text{when } s = 2h \\ \dfrac{2}{N}(-1)^{h}\sum\limits_{j=1}^{m} j^{s}\sin\left[2j\dfrac{\pi(k-p)}{N}\right] & \text{when } s = 2h+1 \end{cases} \tag{21.17}$$

$$p,\ k = 1,\ 2,\ \ldots,\ N, \quad h = 0,\ 1,\ 2,\ \ldots, \quad N = 2m + 1$$

The vector of coefficients $x_{[1]}^{\Gamma}$ of the indefinite integral from the trigonometric polynomial (21.1) with a zero mean ($a_0 = 0$) has the form

$$x_{[1]}^{\Gamma} = \frac{1}{\omega}\left(0,\ -b_1,\ a_1,\ -\frac{1}{2}b_2,\ \frac{1}{2}a_2,\ \ldots,\ -\frac{1}{m}b_m,\ \frac{1}{m}a_m\right) \tag{21.18}$$

The connection between the vectors of coefficients $x_{[1]}^{\Gamma}$ and x^{Γ} is given by the equality

$$x_{[1]}^{\Gamma} = \mu x^{\Gamma} \tag{21.19}$$

where the matrix μ is obtained from the matrix l when $\alpha_0 = 0$, $\alpha_j = 0$, $\beta_j = -\dfrac{1}{j\omega}$, $j = 1,\ 2,\ \ldots,\ m$.

Similarly, for the vector of coefficients $x^{\Gamma}_{[s]}$ of the s-fold indefinite integral from the trigonometric polynomial $x_m(t)$ we have the relation

$$x^{\Gamma}_{[s]} = \mu^s x^{\Gamma} \tag{21.20}$$

where the matrix μ^s has the form of the matrix l with

$$\alpha_0 = 0, \quad \beta_j = 0, \quad \alpha_j = (-1)^h \frac{1}{j^s \omega^s}$$

for even $s (s = 2h)$

$$\alpha_0 = 0, \quad \alpha_j = 0, \quad \beta_j = (-1)^{h+1} \frac{1}{j^s \omega^s}$$

for odd s $(s = 2h + 1)$ (21.21)

The vectors of values $x^{M}_{[1]}$ and $x^{M}_{[s]}$ correspond to the vectors of coefficients $x^{\Gamma}_{[1]}$ and $x^{\Gamma}_{[s]}$, the former being connected with the latter by the formulas

$$x^{M}_{[1]} = M x^{\Gamma}_{[1]} = M\mu x^{\Gamma} = M\mu\Gamma x^{M} = V x^{M}$$

$$x^{M}_{[s]} = M x^{\Gamma}_{[s]} = M\mu^s x^{\Gamma} = M\mu^s\Gamma x^{M} = V^s x^{M}$$

The elements of the matrices $V = \{V^s_{pk}\}$ and $\Theta^s = M\mu^s = \{\Theta^s_{pk}\}$ are obtained from the expressions for the matrices Z and L according to the relations

$$V^s_{pk} = \frac{1}{\omega^s} \begin{cases} \dfrac{2}{N}(-1)^h \displaystyle\sum_{j=1}^{m} \frac{1}{j^s} \cos\left[2j \frac{\pi(p-k)}{N}\right] & \text{when } s = 2h \\ \dfrac{2}{N}(-1)^h \displaystyle\sum_{j=1}^{m} \frac{1}{j^s} \sin\left[2j \frac{\pi(k-p)}{N}\right] & \text{when } s = 2h+1 \end{cases} \tag{21.22}$$

and

$$\Theta_{pk}^{s} = \begin{cases} 0 & \text{when } k = 1 \\ \cos\left[(p-1)k\dfrac{\pi}{N}\right](-1)^{h}\omega^{-s}\left(\dfrac{k}{2}\right)^{-s} & \text{when } k = 2,\ 4,\ldots,N-1 \\ \sin\left[(p-1)k\dfrac{\pi}{N}\right](-1)^{h}\omega^{-s}\left(\dfrac{k-1}{2}\right)^{-s} & \text{when } k = 3,\ 5,\ldots,N \end{cases}$$

for even s $(s = 2h)$,

$$\Theta_{pk}^{s} = \begin{cases} 0 & \text{when } k = 1 \\ \sin\left[(p-1)k\dfrac{\pi}{N}\right](-1)^{h}\left(\dfrac{k}{2}\right)^{-s}\omega^{-s} & \text{when } k = 2,\ 4,\ldots,N-1 \\ \cos\left[(p-1)(k-1)\dfrac{\pi}{N}\right](-1)^{h+1}\left(\dfrac{k-1}{2}\right)^{-s}\omega^{-s} & \text{when } k = 3,\ 5,\ldots,N \end{cases}$$

for odd $s(s = 2h + 1)\, p,\ k = 1, 2,\ldots, N,\ h = 0, 1, 2,\ldots, N = 2m + 1.$

In solving practical problems one has often to deal with the periodic solutions $x^0 = x^0(t)$ which possess symmetry of the form

$$x^0\left(t + \frac{T}{2}\right) = -x^0(t) \tag{21.23}$$

When constructing approximations to such solutions the trigonometric polynomial $x_m(t)$ is taken in the form of a polynomial containing only odd harmonics

$$\overline{x_m}(t) = \sum_{j=1}^{m}{}' (a_j \cos j\omega t + b_j \sin j\psi t) \tag{21.24}$$

where $\sum'$ is summation over odd subscripts.

The formulas for the connection between vectors of values, vectors of coefficients and similar vectors for derivatives and integrals are, in this case, accordingly modified.

Let $\bar{l}$ be a matrix of the form

$$\bar{l} = \begin{bmatrix} \bar{\alpha}_1\bar{\beta}_1 & & & & & 0 \\ -\bar{\beta}_1\bar{\alpha}_1 & & & & & \\ & \bar{\alpha}_3\bar{\beta}_3 & & & & \\ & -\bar{\beta}_3\bar{\alpha}_3 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \bar{\alpha}_m\bar{\beta}_m \\ 0 & & & & & -\bar{\beta}_m\bar{\alpha}_m \end{bmatrix} \tag{21.25}$$

The vector of coefficients of the trigonometric polynomial (21.24) is

$$\bar{x}^\Gamma = (a_1,\ b_1,\ a_3,\ b_3,\ \ldots,\ a_m,\ b_m) \tag{21.26}$$

and its vector of values is

$$\bar{x}^M = (\bar{x}_m(t_0), \bar{x}_m(t_1), \ldots, \bar{x}_m(t_m)) \tag{21.27}$$

Here $(m+1)$ equidistant points are chosen in the interval $\left[0, \frac{T}{2}\right]$ (in view of symmetry)

$$t_i = i\frac{T}{2(m+1)},\quad i = 0, 1, \ldots, m$$

The vectors of coefficients $\bar{x}^\Gamma$ and the vectors of values $\bar{x}^M$ of the trigonometric polynomial $\bar{x}_m(t)$ are in one-to-one correspondence

$$\bar{x}^M = \bar{M}\bar{x}^\Gamma \tag{21.28}$$

$$\bar{x}^\Gamma = \bar{\Gamma}\bar{x}^M \tag{21.29}$$

It is easy to see that the matrices $\bar{M} = \{\bar{M}_{pq}\}$ and $\bar{\Gamma} = \{\bar{\Gamma}_{qp}\}$ are determined by the elements

$$\bar{M}_{pq} = \begin{cases} \cos\left[(p-1)q\dfrac{\pi}{m+1}\right] & \text{when } q = 1, 3, \ldots, m \\ \sin\left[(p-1)(q-1)\dfrac{\pi}{m+1}\right] & \text{when } q = 2, 4, \ldots, m+1 \end{cases}$$

$$p = 1, 2, \ldots, m+1 \tag{21.30}$$

$$\bar{\Gamma}_{pq} = \begin{cases} \dfrac{2}{m+1} \cos\left[p(q-1)\dfrac{\pi}{m+1}\right] & \text{when } p = 1, 3, \ldots, m \\ \dfrac{2}{m+1} \sin\left[(p-1)(q-1)\dfrac{\pi}{m+1}\right] & \text{when } p = 2, 4, \ldots, m+1 \end{cases} \tag{21.31}$$

$$q = 1, 2, \ldots, m+1$$

In the same way we can obtain the elements of the matrices $\bar{Z} = \bar{M}\bar{I} = \{\bar{Z}_{pq}\}$ and $\bar{L} = \bar{M}\bar{I}\bar{\Gamma} = \{\bar{L}_{pq}\}$:

$$\bar{Z}_{pq} = \begin{cases} \cos\left[(p-1)q\dfrac{\pi}{m+1}\right]\bar{\alpha}_q - \sin\left[(p-1)q\dfrac{\pi}{m+1}\right]\bar{\beta}_q \\ \qquad \text{when } q = 1, 3, \ldots, m \\ \cos\left[(p-1)(q-1)\dfrac{\pi}{m+1}\right]\bar{\beta}_{q-1} \\ + \sin\left[(p-1)(q-1)\dfrac{\pi}{m+1}\right]\bar{\alpha}_{q-1} \\ \qquad \text{when } q = 2, 4, \ldots, m+1 \end{cases} \tag{21.32}$$

$$p = 1, 2, \ldots, m+1$$

$$\bar{L}_{pq} = \frac{2}{m+1} \sum_{k=1}^{m} \left(\bar{\alpha}_k \cos\left[k\frac{\pi}{m+1}(p-q)\right] + \bar{\beta}_k \sin\left[k\frac{\pi}{m+1}(q-p)\right] \right)$$

$$p, q = 1, 2, \ldots, m+1$$

The vector of coefficients $\bar{x}^{\Gamma}_{(s)}$ for the sth derivative of the trigonometric polynomial $\bar{x}_m(t)$ and the vector of coefficients

$\bar{x}^{\Gamma}_{(s)}$ for the s-fold indefinite integral from $\bar{x}_m(t)$ are expressed through the vector of coefficients $\bar{x}^{\Gamma}$ as follows

$$\bar{x}^{\Gamma}_{(s)} = \bar{d}^s \bar{x}^{\Gamma} \tag{21.33}$$

$$\bar{x}^{\Gamma}_{[s]} = \bar{\mu}^s \bar{x}^{\Gamma} \tag{21.34}$$

where the matrices $\bar{d}^s$ and $\bar{\mu}^s$ are obtained from the matrix $\bar{l}$ when

$$\bar{\beta}_k = 0, \quad \bar{\alpha}_k = (-1)^h k^s \omega^s \text{ for } s = 2h$$

$$\bar{\alpha}_k = 0, \quad \bar{\beta}_k = (-1)^h k^s \omega^s \text{ for } s = 2h+1 \tag{21.35}$$

and when

$$\bar{\beta}_k = 0, \quad \bar{\alpha}_k = (-1)^h k^{-s} \omega^{-s} \text{ for } s = 2h$$

$$\bar{\alpha}_k = 0, \quad \bar{\beta}_k = (-1)^{h+1} k^{-s} \omega^{-s} \text{ for } s = 2h+1 \tag{21.36}$$

The vector of values $\bar{x}^M_{(s)}$ for the sth derivative of the trigonometric polynomial $\bar{x}_m(t)$ is determined by both the vector of coefficients $\bar{x}^{\Gamma}$ and the vector of values $\bar{x}^M$ according to the following formulas

$$\bar{x}^M_{(s)} = \bar{M}\bar{x}^{\Gamma}_{(s)} = \bar{M}\bar{d}^s\bar{x}^{\Gamma} = \bar{\Phi}^s\bar{x}^{\Gamma}, \qquad \bar{\Phi}^s = \bar{M}\bar{d}^s$$

$$\bar{x}^M_{(s)} = \bar{M}\bar{d}^s\bar{\Gamma}\bar{x}^M = \bar{D}^s\bar{x}^M, \quad D^s = \bar{M}\bar{d}^s\bar{\Gamma}$$

Similarly, for the vector of values $\bar{x}^M_{[s]}$ of the s-fold integral from the polynomial $\bar{x}_m(t)$ we have

$$\bar{x}^M_{[s]} = \overline{M\bar{x}^{\Gamma}_{[s]}} = \overline{M\mu^s}\bar{x}^{\Gamma} = \bar{\Theta}^s\bar{x}^{\Gamma}, \qquad \bar{\Theta}^s = \overline{M\mu^s}$$

$$\bar{x}^M_{[s]} = \overline{M\mu^s}\bar{\Gamma}\bar{x}^M = \bar{V}^s\bar{x}^M, \qquad \bar{V}^s = \overline{M\mu^s}\bar{\Gamma}$$

The elements of the matrices $\bar{\Phi}^s$, $\bar{D}^s$, $\bar{\Theta}^s$, $\bar{N}^s$ are obtained from the expressions for $\bar{Z}$ and $\bar{L}$, taking into account relations (21.35) and (21.36)

$$\bar{\Phi}_{pq} = \begin{cases} \cos\left[(p-1)q\,\dfrac{\pi}{m+1}\right](-1)^h q^s\omega^s \text{ when } q = 1, 3, \ldots, m \\ \sin\left[(p-1)(q-1)\,\dfrac{\pi}{m+1}\right](-1)^h (q-1)^s\omega^s \end{cases}$$

$$\text{when } q = 2, 4, \ldots, m+1 \tag{21.37}$$

$$\bar{\Theta}_{pq} = \begin{cases} \cos\left[(p-1)q\dfrac{\pi}{m+1}\right](-1)^h q^{-s}\omega^{-s} \text{ when } q = 1, 3, \ldots, m \\ \sin\left[(p-1)(q-1)\dfrac{\pi}{m+1}\right](-1)^h (q-1)^{-s}\omega^{-s} \\ \qquad \text{when } q = 2, 4, \ldots, m+1 \end{cases}$$

for even $s = 2h$;

$$\bar{\Phi}^s_{pq} = \begin{cases} \sin\left[(p-1)q\dfrac{\pi}{m+1}\right](-1)^{h+1} q^s\omega^s \text{ when } q = 1, 3, \ldots, m \\ \cos\left[(p-1)(q-1)\dfrac{\pi}{m+1}\right](-1)^h (q-1)^s\omega^s \\ \qquad \text{when } q = 2, 4, \ldots, m+1 \end{cases}$$

$$\bar{\Theta}^s_{pq} = \begin{cases} \sin\left[(p-1)q\dfrac{\pi}{m+1}\right](-1)^h q^{-s}\omega^{-s} \\ \qquad \text{when } q = 1, 3, \ldots, m \\ \cos\left[(p-1)(q-1)\dfrac{\pi}{m+1}\right](-1)^{h+1}(q-1)^{-s}\omega^{-s} \\ \qquad \text{when } q = 2, 4, \ldots, m+1 \end{cases} \tag{21.38}$$

for odd $s = 2h + 1$

$$\bar{D}^s_{pq} = \omega^s \begin{cases} \dfrac{2}{m+1}(-1)^h \sum_{j=1}^{m}{}' j^s \cos\left[j\dfrac{\pi}{m+1}(p-q)\right] \\ \qquad \text{when } s = 2h \\ \dfrac{2}{m+1}(-1)^h \sum_{j=1}^{m}{}' j^s \sin\left[j\dfrac{\pi}{m+1}(q-p)\right] \\ \qquad \text{when } s = 2h+1 \end{cases} \tag{21.39}$$

$$\bar{V}^s_{pq} = \omega^{-s} \begin{cases} \dfrac{2}{m+1}(-1)^h \sum\limits_{j=1}^{m}{}' j^{-s} \cos\left[j \dfrac{\pi}{m+1}(p-q)\right] \\ \qquad\qquad \text{when } s = 2h \\ \dfrac{2}{m+1}(-1)^h \sum\limits_{j=1}^{m}{}' j^{-s} \sin\left[j \dfrac{\pi}{m+1}(q-p)\right] \\ \qquad\qquad \text{when } s = 2h+1 \end{cases} \tag{21.40}$$

$$p,\ q = 1,\ 2, \ldots, m+1, \quad h = 0,\ 1,\ 2, \ldots$$

Note some peculiarities of the transformation matrices introduced above. It is evident that the matrices D^s, V^s, $\bar{D}^s$, and $\bar{V}^s$ are symmetric when s is even and skew-symmetric when it is odd. They have the same elements on their diagonals (along the main diagonal). In fact, if in the expressions for their elements we replace the indices p and k by $(p+1)$ and $(k+1)$ respectively, we shall have

$$D^s_{pk} = D^s_{p+1,k+1}, \quad V^s_{pk} = V^s_{p+1,k+1}$$

for all $p,\ k = 1,\ 2, \ldots, 2m+1$, $s = 1,\ 2, \ldots$; and

$$\bar{D}^s_{pk} = \bar{D}^s_{p+1,k+1}, \quad \bar{V}_{pk} = \bar{V}_{p+1,k+1}$$

for all $p,\ k = 1,\ 2, \ldots, m+1$, $s = 1,\ 2, \ldots$.

Further, from formulas (21.17) and (21.22) which express the elements of the matrices D^s and V^s we have

$$D^s_{p,\,2m+1} = D^s_{p+1,\,1}, \quad V^s_{p,\,2m+1} = V^s_{p+1,\,1}$$

for all $p = 1,\ 2, \ldots, 2m+1$, $s = 1,\ 2, \ldots$.

Hence, to obtain the elements of the second row of the matrices D^s and V^s from the corresponding elements of their first rows, one should put the last element of the first row before the first one. Similarly, it follows from the equalities (21.39) and (21.40) which determine the elements of the matrices $\bar{D}^s$ and $\bar{V}^s$ that

$$\bar{D}^s_{p,m+1} = -\bar{D}^s_{p+1,1}, \quad \bar{V}^s_{p,m+1} = -\bar{V}^s_{p+1,1}, \quad p = 1, 2, \ldots$$

i. e. to obtain the elements of the second row of the matrices $\bar{D}^s$ and $\bar{V}^s$ from the elements of their first rows, the last element of the first row is taken with the opposite sign. Thus any next row is obtained from the elements of the preceding one. When using these matrices, therefore, there is no need to calculate all the matrices and store them in digital computer memory in full: it is enough to calculate and store only their first rows.

The concept of a vector of values has been introduced for the periodic function of period T, which is a trigonometric polynomial. Quite analogously this concept could be extended to include the arbitrary periodic function of period T.

In practice it is convenient to use the vector of values of trigonometric polynomials owing to the following properties.

1. Let

$$x_m(t) = a_0 + \sum_{j=1}^{m} (a_j \cos j\omega t + b_j \sin j\omega t)$$

and

$$y_m(t) = a_0 + \sum_{j=1}^{m} (c_j \cos j\omega t + d_j \sin j\omega t)$$

be two trigonometric polynomials of order m. Then the vector of values z^M of the trigonometric polynomial equal to the sum they make $z_m(t) = x_m(t) + y_m(t)$ will be the sum of their vectors of values $z^M = x^M + y^M$.

2. Let $z(t)$ be the product of the trigonometric polynomials $z(t) = x_m(t)\, y_m(t)$. Then the vector of values z^M of the trigonometric polynomial $z(t)$ is obtained by component-wise multiplication of the vectors x^M and y^M: $z^M = x^M y^M$.

3. *Let* $z(t) = (x_m(t))^k$ and let k be a rational number. Then the vector of values z^M of the function $z(t)$ is found by raising component-wise the vector of values x^M to the kth power.

4. Let $z(t)$ be a continuous function from the trigonometric polynomial $x_m(t)$ and its derivatives

$$z(t) = f\left(x_m, \frac{dx_m}{dt}, \frac{d^2x_m}{dt}, \ldots\right)$$

Then the vector of values z^M for the function $z(t)$ is the vector $z^M = f(x^M, \dot{D}x^M, D^2x^M, \ldots) = f^M$ which is obtained by a component-wise action of the function f on the corresponding vector argument.

22. FORMALIZATION OF THE PROCESS OF SETTING UP ALGEBRAIC EQUATIONS OF THE TRIGONOMETRIC COLLOCATION METHOD

The results of the foregoing section have laid a foundation for formalizing the process of obtaining and solving a system of equations of the collocation method, which is convenient for practical realization on digital computers.

The concept of vectors of coefficients and values of a trigonometric polynomial can be extended in the natural way to include vectorial trigonometric polynomials. In what follows, for the vectorial trigonometric polynomial

$$x_m(t) = (x_{m1}(t), \ldots, x_{mn}(t)) \tag{22.1}$$

by x^Γ and x^M we shall understand the vector of coefficients and the vector of values

$$x^\Gamma = (x_1^\Gamma, x_2^\Gamma, \ldots, x_n^\Gamma) \tag{22.2}$$

$$x^M = (x_1^M, x_2^M, \ldots, x_n^M) \tag{22.3}$$

According to the trigonometric collocation method, constructing an approximate periodic solution of the system of differential equations (18.1) requires determining unknown coefficients of the polynomial (19.2) from the system of equations (19.5).

Using the vectors of coefficients and the matrices of transformations over them, which were introduced above, it is easy to expand the system of algebraic equations (19.5) by writing it in the form

$$\begin{aligned}
\Phi^1 x_1^\Gamma &= F_1(t_i, Mx_1^\Gamma(i), Mx_2^\Gamma(i), \ldots, Mx_n^\Gamma(i)) \\
\Phi^1 x_2^\Gamma &= F_2(t_i, Mx_1^\Gamma(i), Mx_2^\Gamma(i), \ldots, Mx_n^\Gamma(i)) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\Phi^1 x_n^\Gamma &= F_n(t_i, Mx_1^\Gamma(i), Mx_2^\Gamma(i), \ldots, Mx_n^\Gamma(i))
\end{aligned} \tag{22.4}$$

Here the vector $F_k(t_i, Mx_1^\Gamma(i), \ldots, Mx_n^\Gamma(i)) = F_k(t_i, Mx^\Gamma(i))$ has the form

$$\begin{aligned}
F_k(t_i, Mx^\Gamma(i)) = (&f_k(t_0, Mx_1^\Gamma(1), \ldots, Mx_n^\Gamma(1)) \\
&f_k(t_1, Mx_1^\Gamma(2), \ldots, Mx_n^\Gamma(2)), \\
&\ldots, f_k(t_{2m}, Mx_1^\Gamma(2m+1), \ldots, Mx_n^\Gamma(2m+1))
\end{aligned} \tag{22.5}$$

the index i in round brackets following the vector shows that an ith coordinate of this vector is taken; thus $Mx_j^\Gamma(i)$ is the ith component of the vector $Mx_j^\Gamma = x_j^M$; the matrix Φ^1 is a matrix of the form (21.17) with $h = 0,\ k,\ j = 1,\ 2, \ldots, n$.

In a special case when a periodic solution of a single differential equation of nth order (19.22) is sought, the system of algebraic equations (19.24) is expanded as follows

$$\Phi^n x^\Gamma + \operatorname{diag}[c_1(t_i)]\,\Phi^{n-1}x^\Gamma + \ldots + \operatorname{diag}[c_n(t_i)]\,Mx^\Gamma - \Psi(t_i, Mx^\Gamma(i), \Phi^1 x^\Gamma(i), \ldots, \Phi^{n-1}x^\Gamma(i)) = f^M \quad (22.6)$$

where

$$\Psi(t_i, Mx^\Gamma(i), \Phi^1 x^\Gamma(i), \ldots, \Phi^{n-1}x^\Gamma(i))$$

$$= (\psi(t_0, Mx^\Gamma(1), \Phi^1 x^\Gamma(1), \ldots, \Phi^{n-1}x^\Gamma(1))$$

$$\psi(t_1, Mx^\Gamma(2), \Phi^1 x^\Gamma(2), \ldots, \Phi^{n-1}x^\Gamma(2)),$$

$$\ldots, \psi(t_{2m}, Mx^\Gamma(2m+1), \Phi^1 x^\Gamma(2m+1), \ldots, \Phi^{n-1}x^\Gamma(2m+1)) \quad (22.7)$$

$$f^M = (f(t_0), f(t_1), \ldots, f(t_{2m}))$$

$$\operatorname{diag}[c_j(t_i)] = \operatorname{diag}\{c_j(t_0), c_j(t_1), \ldots, c_j(t_{2m})\} \quad (22.8)$$

is a diagonal matrix.

For the linear system of equations (19.26) the algebraic equations (22.4) have the form (19.27) since Φ^1 coincides with the matrix M'.

Equations (22.4), (22.6) and (19.27) are formalized in such a way that they can be easily programmed on a digital computer. To solve them one can make effective use of such methods as the Newton-Kantorovich method [60], the method of continuing the solution by a parameter [31], a modification of the method of steepest descent [111], various versions of the method of simple iteration [148, 149] the numerical realization of which on a digital computer is standardized and well developed. On carrying out all the calculations required we obtain approximate values of coefficients of a trigonometric polynomial (19.2) which approximates the exact periodic solution.

For many practical purposes it is the values of an approximation that are important rather than its analytical form. In view of

this it is convenient and useful to modify the scheme of the trigonometric collocation method to an extent which would allow one to obtain directly algebraic equations for these values. This is easy to do by taking advantage of matrices of transition from vectors of coefficients to vectors of values of trigonometric polynomials.

The system of equations (19.5) takes the following form with respect to the vector of values x^M

$$D^1 x_1^M = F_1(t_i, x_1^M(i), x_2^M(i) \ldots, x_n^M(i))$$

$$D^1 x_2^M = F_2(t_i, x_1^M(i), x_2^M(i), \ldots, x_n^M(i))$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$D^1 x_n^M = F_n(t_i, x_1^M(i), x_2^M(i), \ldots, x_n^M(i)) \tag{22.9}$$

where

$$F_k(t_i, x_1^M(i), x_2^M(i), \ldots, x_n^M(i)) = F_k(t_i, x^M(i))$$

$$= (f_k(t_0, x^M(1)), f_k(t_1, x^M(2)), \ldots, f_k(t_{2m}, x^M(2m+1))) \tag{22.10}$$

$$x^M(i) = (x_1^M(i), x_2^M(i), \ldots, x_n^M(i)), k = 1, 2, \ldots, n \tag{22.11}$$

and D^1 is a matrix of the form (21.17) with $h = 0$.

Similarly, for a differential equation of nth order and a system of linear differential equations the algebraic equations (19.24) and (19.27) take the following form with respect to vectors of values

$$D^n x^M + \operatorname{diag}[c_1(t_i)] D^{n-1} x^M + \ldots + \operatorname{diag}[c_n(t_i)] x^M$$

$$- \Psi(t_i, x^M(i), D^1 x^M(i), \ldots, D^{n-1} x^M(i)) = f^M \tag{22.12}$$

$$\begin{bmatrix} \tilde{h}_{11} & \tilde{h}_{12} & \ldots & \tilde{h}_{1n} \\ \tilde{h}_{21} & \tilde{h}_{22} & \ldots & \tilde{h}_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \tilde{h}_{n1} & \tilde{h}_{n2} & \ldots & \tilde{h}_{nn} \end{bmatrix} \begin{bmatrix} x_1^M \\ x_2^M \\ \ldots \\ \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \ldots \\ F_n \end{bmatrix} \tag{22.13}$$

where

$$\Psi(t_i, x^M(i), D^1x^M(i), \ldots, D^{n-1}x^M(i))$$

$$= (\psi(t_0, x^M(1), D^1x^M(1), \ldots, D^{n-1}x^M(1)),$$
$$\ldots, \psi(t_{2m}, x^M(2m+1), D^1x^M(2m+1), \ldots, D^{n-1}x^M(2m+1))) \tag{22.14}$$

$$\tilde{h}_{pq} = \operatorname{diag}[a_{pq}(t_k)] \text{ when } p \neq q$$

$$\tilde{h}_{pp} = \operatorname{diag}[a_{pp}(t_k)] - D^1 \tag{22.15}$$

F_j is the vector (19.28); $p, q = 1, 2, \ldots, n$; $k = 0, 1, 2, \ldots, 2m$.

Comparing the corresponding equations for x^Γ and x^M convinces one that in terms of the number of operations it is more advantageous to carry out direct calculations by considering the equations of the trigonometric collocation method with respect to the vectors of values.

In the case when the periodic solution to be sought possesses symmetry of the form (21.23) the algebraic equations of the trigonometric collocation method are obtained from (22.4), (22.6), (22.9) and (22.12) by replacing the matrices Φ^s and D^s by the matrices $\tilde{\Phi}^s$ and $\tilde{D}^s$, respectively.

23. EXAMPLES

To illustrate the procedure developed above for calculating periodic solutions of differential equations, we shall now give some particular examples.

EXAMPLE 23.1. Section 10 gives a general equation for torsional oscillations of shaft consisting of two masses joined by means of a nonlinear elastic connection one of which is acted upon by a periodic torsional moment $F(t)$. Let us calculate these oscillations for several particular values of parameters and functions contained in this equation.

Put the following in equation (10.45)

$$J_1 = J_2 = 1, \qquad c(x) = \frac{1}{2}x + 1000x^3$$

$$f\left(\frac{dx}{dt}\right) = 5\frac{dx}{dt} + \left(\frac{dx}{dt}\right)^3 \tag{23.1}$$

$$F(t) = 10\sin 10t + A\sin 50t$$

and make calculations for cases where A is equal in value to 90, 60 and 40. In the case of equation (10.45) we are considering, the nonlinear terms are odd functions and the torsional moment $F(t)$ is determined by odd harmonics. The approximate periodic solution of period $T = \frac{\pi}{5}$ of the equation is, therefore, sought in the form (21.24).

Take 24 equidistant points in the interval $\left[0, \frac{\pi}{10}\right]$, i. e. put $m = 23$.

The system of algebraic equations of the trigonometric collocation method has the form

$$(\bar{D}^2 + 5\bar{D}^1 + E)\bar{x}^M = -(\bar{D}^1\bar{x}^M)^3 - 2000(\bar{x}^M)^3 + F^M \quad (23.2)$$

where the vector $F^M = \bar{M}F^\Gamma$;

$$F^\Gamma = (0,\ 10,\ 0,\ 0,\ 0,\ A,\ 0,\ 0, \ldots, 0)$$

$$\bar{M} = \{\bar{M}_{pq}\}, \quad \bar{D}^s = \{\bar{D}^s_{pq}\} \quad (s = 1,\ 2)$$

$$\bar{M}_{pq} = \begin{cases} \cos\left[(p-1)q\dfrac{\pi}{24}\right] & \text{when } q = 1, 3, \ldots, 23 \\ \sin\left[(p-1)(q-1)\dfrac{\pi}{24}\right] & \text{when } q = 2, 4, \ldots, 24 \end{cases} \quad (23.3)$$

$$\bar{D}^s_{pq} = 10^s \begin{cases} \dfrac{1}{12}(-1)^h \sum\limits_{j=1}^{23}{}' j^s \cos\left[j\dfrac{\pi}{24}(p-q)\right] & \text{when } s = 2h \\ \dfrac{1}{12}(-1)^h \sum\limits_{j=1}^{23}{}' j^s \sin\left[j\dfrac{\pi}{24}(q-p)\right] & \text{when } s = 2h+1 \end{cases} \quad (23.4)$$

$$p,\ q = 1,\ 2, \ldots, 24, \quad h = 0,\ 1$$

The roots of the system of equations (23.2) are found by the

method of simple iteration

$$\bar{x}^M_{\{\nu+1\}} = (\bar{D}^2 + 5\,\bar{D}^1 + E)^{-1}\,[-\bar{D}^1(\bar{x}^M_{\{\nu\}})^3 - 2000(\bar{x}^M_{\{\nu\}})^3 + F^M] \tag{23.5}$$

$$\nu = 0,\ 1,\ 2, \ldots$$

where the indices in braces indicate the number of iterations.

For $A = 90$ we take as the initial approximation $\bar{x}^M_{\{0\}}$ the vector composed of values of the function

$$x_{\{0\}}(t) = -0.06\cos 10t - 0.07\sin 10t \tag{23.6}$$

which is an approximate solution of a linearized equation obtained taking into account only the fundamental harmonic.

So $\bar{x}^M_{\{0\}} = \bar{M}\bar{x}^\Gamma_{\{0\}}$, where $\bar{x}^\Gamma_{\{0\}} = (-0.06;\ -0.07;\ 0,\ 0, \ldots, 0)$.

Curve *II* of Figure 4 represents the initial approximation. The Euclidean norm of the discrepancy vector $P(\bar{x}^M)$ for the initial approximation $\bar{x}^M = \bar{x}^M_{\{0\}}$ has the value $\| P(\bar{x}^M_{\{0\}}) \| = 90.54$. For this discrepancy the iteration scheme (23.5) gives the value $\| p(\bar{x}_{\{12\}}) \| = 0.24 \cdot 10^{-1}$ at the 12th step and the value $\| P(\bar{x}^M_{\{24\}}) \| = 0.13 \cdot 10^{-4}$ at the 24th. The vector of values $\bar{x}^M_{\{24\}}$ is represented by curve *I* of Figure 4. The curve has been obtained assuming the initial approximation determined by the values of the function (23.6). Since, generally speaking, the choice of the initial approximation allows sufficiently large arbitrariness, it is natural to calculate the periodic solution assuming an initial approximation substantially different from that already considered so that the results obtained may be compared.

If one takes the vector $\bar{x}^M_{\{0\}}$ composed of the values of the function

$$x_{\{0\}}(t) = -0.12\cos 10t - 0.14\sin 10t \tag{23.7}$$

(they are represented by curve *III* of Figure 4) as the initial approximation, then $\| P(\bar{x}^M_{\{0\}}) \| = 99.85$; at the 24th step the discrepancy is estimated to have the value $\| P(\bar{x}^M_{\{24\}}) \| = 0.5 \cdot 10^{-4}$, the vector of values of the approximate periodic solution coincides with curve *I* of Figure 4 to within seven decimal points. This shows good approximation of a periodic solution if there is sufficient degree of arbitrariness in the choice of an initial approximation.

The periodic solution of the equation under consideration was also constructed by us by the Runge-Kutta fourth-order numerical method as the solution of the initial-value problem with the initial

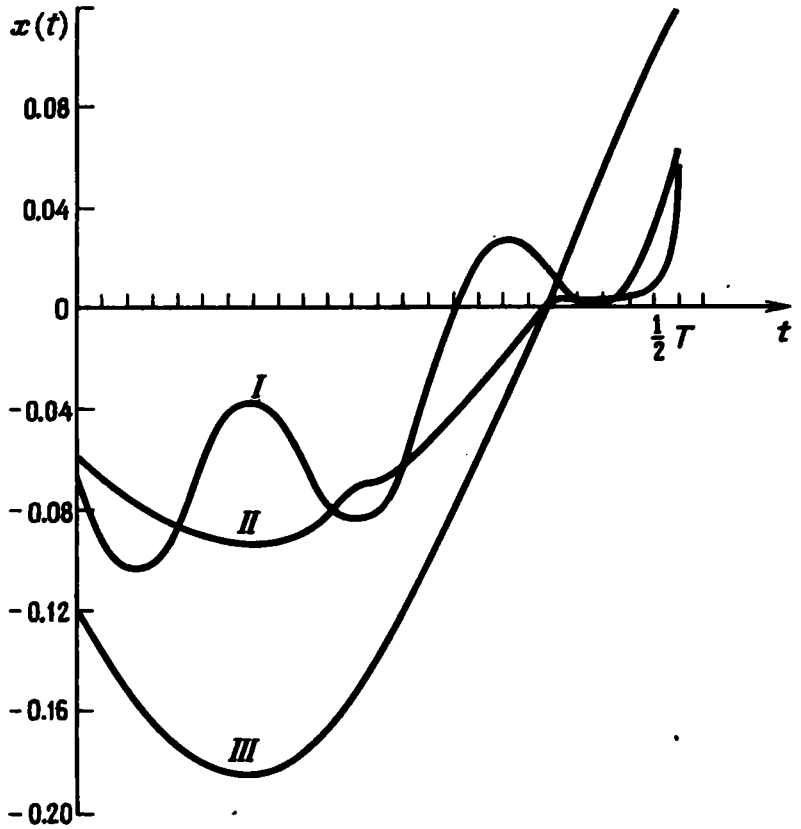

Fig. 4

values of the periodic solution. The initial values of such a solution were found as the approximate values of the fixed point of the trajectory displacement operator calculated by the method of simple iteration assuming the initial approximation $t = 0$, $x(0) = 0$, $dx(0)/dt = 0$. The numerical integration was carried out in the interval $[0, 16T] = [0, 16\pi/5]$ with the step $h = T/200 = \pi/1000$. The results of the calculation coincided well (to within the sixth decimal point) with the values calculated by the trigonometric collocation method.

For $A = 60$ the periodic solution was sought by the same method but starting from the initial approximation composed of the values of the function

$$x_{\{0\}}(t) = -0.038\cos 10t - 0.115\sin 10t + 0.003\cos 30t + 0.0007\sin 30t + 0.00005\cos 50t - 0.016\sin 50t \tag{23.8}$$

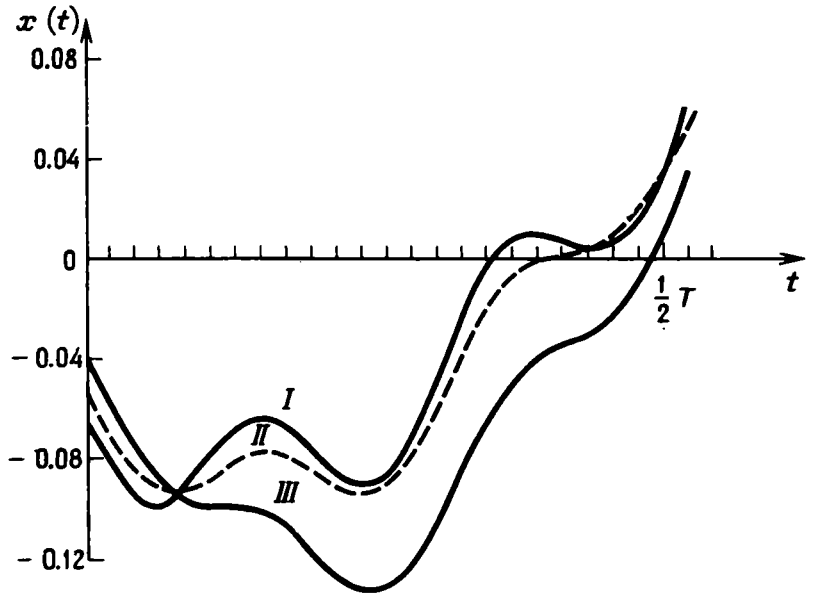

Fig. 5

and

$$x_{\{0\}}(t) = -0.042\cos 10t - 0.15\sin 10t + 0.008\cos 30t$$
$$+ 0.00009\sin 30t + 0.00001\cos 50t - 0.025\sin 50t \qquad (23.9)$$

The initial values for the function (23.8) are represented by curve *III* of Figure 5. The initial discrepancy was equal to the value $\| P(\bar{x}^M_{\{0\}}) \| = 27.13$ in the first case and $\| P(\bar{x}^M_{\{0\}}) \| = 37.17$ in the second. At the 20th step of the iteration process (23.5) $\| P(\bar{x}^M_{\{20\}}) \| = 0.15 \cdot 10^{-6}$ in the first case and $\| P(\bar{x}^M_{\{12\}}) \| = 0.67 \cdot 10^{-3}$ at the 12th step in the second case.

Curve *I* of Figure 5 represents the values of $\bar{x}^M_{\{20\}}$. The vector of values $\bar{x}^M_{\{12\}}$ coincided with the vector of values $\bar{x}^M_{\{20\}}$ to within six decimal points.

From the numerical values of $\bar{x}^M_{\{20\}}$ the vector of coefficients of the trigonometric polynomial, of the approximate solution, was calculated. Its coordinates $a_1, b_1, \ldots, a_{23}, b_{23}$ are given in the left column of Table 2.

For $A = 40$ the periodic solution was sought proceeding from the initial approximation $\bar{x}^M_{\{0\}}$ determined from the values of the function (23.8). Here $\| P(\bar{x}^M_{\{0\}}) \| = 15.87$, while at the 12th step

$\| P(\bar{x}^M_{\{0\}}) \| = 0.13 \cdot 10^{-4}$ was obtained. A periodic solution (curve *II* of Figure 5) was constructed by the calculated values of $\bar{x}^M_{\{12\}}$ which proved "close" to the periodic solution of equation (10.45) for $A = 60$. The coordinates of the vector of coefficients $\bar{x}^\Gamma_{\{12\}}$ calculated from the vector of values $\bar{x}^M_{\{12\}}$ are given in the right-hand column of Table 2.

TABLE 2

$\bar{x}^\Gamma_{\{20\}}$	$\bar{x}^\Gamma_{\{12\}}$
$-0.581\,240\,509 \cdot 10^{-1}$	$-0.568\,789\,393 \cdot 10^{-1}$
$-0.644\,682\,184 \cdot 10^{-1}$	$-0.732\,632\,573 \cdot 10^{-1}$
$-0.295\,564\,018 \cdot 10^{-3}$	$-0.439\,548\,410 \cdot 10^{-4}$
$-0.653\,802\,210 \cdot 10^{-3}$	$-0.416\,276\,680 \cdot 10^{-3}$
$-0.344\,360\,066 \cdot 10^{-2}$	$-0.216\,877\,852 \cdot 10^{-2}$
$-0.237\,182\,642 \cdot 10^{-1}$	$-0.158\,700\,163 \cdot 10^{-1}$
$-0.662\,163\,029 \cdot 10^{-4}$	$0.374\,959\,754 \cdot 10^{-4}$
$0.125\,567\,169 \cdot 10^{-3}$	$0.105\,073\,867 \cdot 10^{-3}$
$-0.105\,589\,788 \cdot 10^{-3}$	$-0.521\,813\,103 \cdot 10^{-4}$
$-0.437\,470\,874 \cdot 10^{-4}$	$-0.194\,612\,720 \cdot 10^{-4}$
$-0.276\,638\,380 \cdot 10^{-4}$	$-0.164\,882\,329 \cdot 10^{-4}$
$0.718\,996\,515 \cdot 10^{-4}$	$0.328\,279\,853 \cdot 10^{-4}$
$-0.161\,773\,005 \cdot 10^{-5}$	$-0.354\,000\,008 \cdot 10^{-6}$
$-0.448\,225\,947 \cdot 10^{-6}$	$-0.220\,164\,479 \cdot 10^{-6}$
$-0.159\,614\,534 \cdot 10^{-4}$	$-0.483\,455\,949 \cdot 10^{-6}$
$0.102\,562\,587 \cdot 10^{-4}$	$0.296\,859\,255 \cdot 10^{-6}$
$0.662\,743\,332 \cdot 10^{-6}$	$0.257\,678\,152 \cdot 10^{-6}$
$-0.537\,396\,742 \cdot 10^{-6}$	$-0.149\,038\,599 \cdot 10^{-6}$
$0.950\,296\,908 \cdot 10^{-8}$	$0.498\,062\,036 \cdot 10^{-9}$
$-0.435\,970\,280 \cdot 10^{-8}$	$0.966\,592\,623 \cdot 10^{-7}$
$0.298\,086\,774 \cdot 10^{-6}$	$0.636\,664\,481 \cdot 10^{-7}$
$0.253\,887\,178 \cdot 10^{-8}$	$0.738\,765\,031 \cdot 10^{-8}$
$0.471\,807\,198 \cdot 10^{-7}$	$0.485\,465\,534 \cdot 10^{-8}$
$-0.238\,529\,085 \cdot 10^{-7}$	$-0.359\,965\,571 \cdot 10^{-8}$

The system of algebraic equations (23.2) was also solved using the scheme of the method of simple iteration

$$\bar{x}^M_{\{\nu+1\}} = \bar{V}^2 \left[(-5\,\bar{D}^1 - E)\bar{x}^M_{\{\nu\}} - (\bar{D}^1 \bar{x}^M_\nu)^3 - 2000(\bar{x}^M_\nu)^3 + F^M \right] \tag{23.10}$$

where $\bar{V}^2 = \{ \bar{V}^2_{pq} \}$;

$$\bar{V}^2_{pq} = -\frac{10^{-2}}{12} \sum_{j=1}^{23\prime} j^{-2} \cos\left[j\frac{\pi}{24}(p-q)\right] \qquad (23.11)$$

$$p,\ q = 1,\ 2, \ldots, 24$$

Calculations have shown that the convergence of approximations by the iteration formula (23.10) is slower than by (23.5). To obtain similar values of periodic solutions in this case requires five to seven iterations more.

EXAMPLE 23.2. Let us determine the periodic motion of a nonlinear vibrator of period $T = 2\pi/10 = \pi/5$ which is influenced by a superposed sinusoidal force, the equation for the oscillations of the vibrator having the form

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + x = \left(\frac{dx}{dt}\right)^3 - 2000\,x^3 + 10\sin 10t \qquad (23.12)$$

This solution is found from formula (21.24) by disjoining the half interval first into six parts ($m = 5$) and then into 24 ($m = 23$).

The system of algebraic equations of the trigonometric collocation method for equation (23.12) with respect to the vector of values $\bar{x}^M$ has the form

$$(\bar{D}^2 + 5\bar{D}^1 + E)\bar{x}^M = (\bar{D}^1\bar{x}^M)^3 - 2000(\bar{x}^M)^3 + f^M \qquad (23.13)$$

where f^M is the vector of values for the function $10\sin 10t$. To determine the roots of this system of equations one can use the method of simple iteration

$$\bar{x}^M_{\{\nu+1\}} = (\bar{D}^2 + 5\bar{D}^1 + E)^{-1}\left[(\bar{D}^1\bar{x}^M_{\{\nu\}})^3 - 2000(\bar{x}^M_{\{\nu\}})^3 + f^M\right] \qquad (23.14)$$

For $m = 5$, as the initial vector of values $\bar{x}^M_{\{0\}}$ we take the vector

$$\bar{x}^M_{\{0\}} = \begin{bmatrix} -0.003\ 790 \\ -0.017\ 010 \\ -0.085\ 010 \\ -0.072\ 000 \\ -0.063\ 100 \\ -0.002\ 120 \end{bmatrix}$$

equal to the approximate values at the collocation points of the function which is a solution of a linearized equation. Curve *I* (Figure 6) represents the values of that vector.

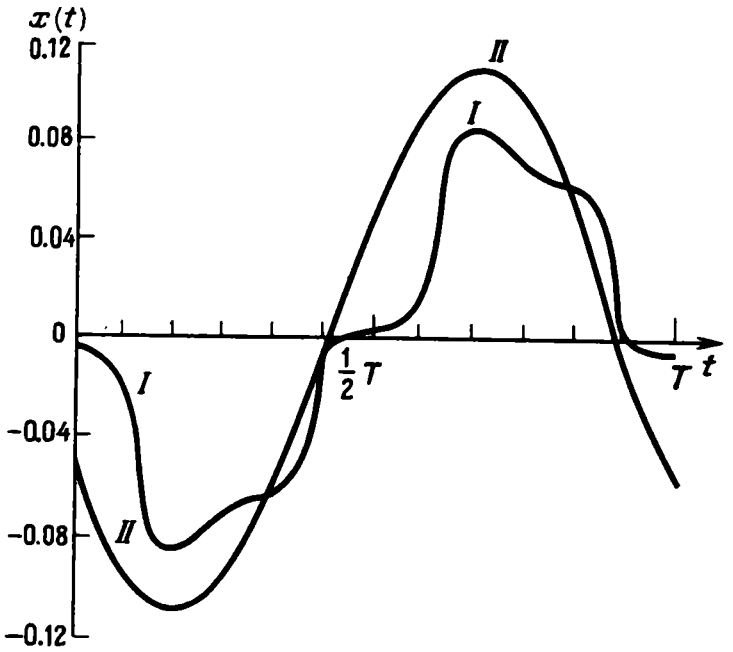

Fig. 6

For $P(\bar{x}^M_{\{0\}})$ we have $\| P(\bar{x}^M_{\{0\}}) \| = 72.69$.

At the 10th step the iteration scheme (23.14) yields

$$\bar{x}^M_{\{10\}} = \begin{bmatrix} -0.491\ 747\ 111 \cdot 10^{-1} \\ -0.926\ 832\ 640 \cdot 10^{-1} \\ -0.110\ 747\ 587 \\ -0.977\ 779\ 174 \cdot 10^{-1} \\ -0.592\ 322\ 954 \cdot 10^{-1} \\ -0.613\ 140\ 583 \cdot 10^{-2} \end{bmatrix} \tag{23.15}$$

and the discrepancy $\| P(\bar{x}^M_{\{10\}}) \| = 0.8 \cdot 10^{-2}$.

The values of $\bar{x}^M_{\{10\}}$ are represented by curve *II* (Figure 6).

By the 24 points, starting from the initial approximation (23.8), at the 12th step we have

$$\begin{bmatrix} -0.493\ 156\ 827 \times 10^{-1*} \\ -0.619\ 221\ 078 \cdot 10^{-1} \\ -0.755\ 460\ 41 \cdot 10^{-1} \\ -0.839\ 749\ 302 \cdot 10^{-1} \\ -0.929\ 573\ 911 \cdot 10^{-1*} \\ -0.100\ 300\ 546 \\ -0.105\ 838\ 800 \\ -0.109\ 452\ 024 \end{bmatrix}$$

$$\bar{x}^M_{\{12\}} = \begin{bmatrix} -0.111\,071\,698^* \\ -0.110\,682\,745 \\ -0.108\,321\,119 \\ -0.104\,067\,924 \\ -0.980\,413\,366\cdot 10^{-1*} \\ -0.903\,876\,318\cdot 10^{-1} \\ -0.812\,725\,410\cdot 10^{-1} \\ -0.708\,737\,973\cdot 10^{-1} \\ -0.593\,754\,470\cdot 10^{-1*} \\ -0.469\,641\,749\cdot 10^{-1} \\ -0.338\,276\,469\cdot 10^{-1} \\ -0.201\,546\,693\cdot 10^{-1} \\ -0.613\,678\,920\cdot 10^{-2*} \\ -0.802\,918\,998\cdot 10^{-2} \\ 0.221\,384\,500\cdot 10^{-1} \\ 0.359\,759\,561\cdot 10^{-1} \end{bmatrix}$$

Asterisks denote the values of the approximate solution at the points of the values (23.15). Here $\| P(\bar{x}^M_{\{0\}}) \| = 41.13$; $\| P(\bar{x}^M_{\{12\}}) \| = 0.2\cdot 10^{-2}$, and differences between the values at the common points of the approximations $\bar{x}^M_{\{10\}}$ and $\bar{x}^M_{\{12\}}$ are observed at the third decimal point.

EXAMPLE 23.3. Let us consider the equation

$$\frac{d^2x}{dt^2} + 0.2\cos t x^2 = f(t) \tag{23.16}$$

where $f(t) = -0.975\cos t - 0.398\sin 2t + 0.001\sin 4t - 0.0005\cos 5t$. An approximate periodic solution of period 2π of this equation is sought in the form of the trigonometric polynomial (19.23). The system of algebraic equations with respect to the vector x^M has the following form

$$D^2x^M + 0.2\,\mathrm{diag}\,[\cos t_i](x^M)^2 = f^M \tag{23.17}$$

where $f^M = Mf^\Gamma$; $f^\Gamma = (0;\ -0.975;\ 0;\ 0;\ 0.398;\ 0;\ 0;\ 0;\ 0.001;\ -0.0005;\ 0;\ 0,\ldots,0)$. The roots of the system have been found for $m = 8$, starting from the initial vector corresponding to the zero approximation

$$x_{\{0\}}(t) = 0.07\sin 2t \tag{23.18}$$

Here $\| P(x^M_{\{0\}}) \| = 63.98$; the discrepancy $P(x^M_{\{7\}}$ found by $x^M_{\{7\}}$ is by the norm equal to $\| P(x^M_{\{7\}}) \| = 0.1\cdot 10^{-7}$. The components of the vectors of values $x^M_{\{7\}}$ and that of coefficients a_0, a_1,

$b_1, \ldots, a_8, b_8$ calculated using the iteration formula

$$x^M_{\{\nu+1\}} = V^2 [f^M - \operatorname{diag}[0.2 \cos t_i](x^M_{\{\nu\}})^2] \tag{23.19}$$

are given in Table 3.

TABLE 3

$x^M_{\{7\}}$	$x^r_{\{7\}}$
$0.999\,759\,504 \times 10^{-1}$	$0.363\,797\,881 \cdot 10^{-11}$
$0.160\,623\,403$	$0.100\,000\,001$
$0.173\,494\,816$	$0.304\,672\,641 \cdot 10^{-11}$
$0.124\,357\,863$	$-0.545\,696\,821 \cdot 10^{-11}$
$0.275\,910\,965 \cdot 10^{-1}$	$0.100\,000\,000$
$-0.799\,859\,445 \cdot 10^{-1}$	$-0.268\,014\,446 \cdot 10^{-7}$
$-0.156\,448\,316$	$-0.138\,953\,555 \cdot 10^{-11}$
$-0.174\,560\,518$	$-0.227\,373\,675 \cdot 10^{-11}$
$-0.134\,407\,001$	$0.150\,563\,411 \cdot 10^{-7}$
$-0.621\,586\,235 \cdot 10^{-1}$	$-0.240\,132\,648 \cdot 10^{-7}$
$0.447\,228\,198 \cdot 10^{-2}$	$-0.227\,291\,631 \cdot 10^{-11}$
$0.359\,169\,148 \cdot 10^{-1}$	$0.159\,161\,573 \cdot 10^{-11}$
$0.253\,005\,175 \cdot 10^{-1}$	$-0.667\,467\,493 \cdot 10^{-8}$
$-0.915\,881\,642 \cdot 10^{-2}$	$-0.490\,717\,866 \cdot 10^{-8}$
$-0.352\,456\,405 \cdot 10^{-1}$	$0.664\,024\,934 \cdot 10^{-11}$
$-0.256\,521\,601 \cdot 10^{-1}$	$0.272\,848\,410 \cdot 10^{-11}$
$0.258\,841\,755 \cdot 10^{-1}$	$-0.375\,787\,950 \cdot 10^{-8}$

The exact periodic solution of equation (23.16) is

$$x^0(t) = 0.1 \cos t + 0.1 \sin 2t \tag{23.20}$$

As can be seen from the table, the coefficients of the approximate solution that has been found coincide with those of the exact solution up to the eighth decimal point.

EXAMPLE 23.4. Find the periodic solution of period $2\pi/\omega$ of the equation

$$\frac{dx}{dt} + \left(\frac{d^2x}{dt^2}\right)^3 + 3x^3 = A \sin \omega t + B \sin 3\omega t + C \sin 5\omega t \tag{23.21}$$

for various values of the frequency ω and the amplitudes A, B and C.

With respect to the vector of values $\bar{x}^M$ of the approximate periodic solution we have the system of equations

$$D^1 \bar{x}^M + (\bar{D}^2 \bar{x}^M)^3 + 3(\bar{x}^M)^3 = f^M \tag{23.22}$$

where f^M is the vector of values of the function $A\sin\omega t + B\sin 3\omega t + C\sin 5\omega t$.

Choose $m = 5$ and, to solve the system of equations (23.22), use the iteration scheme

$$\bar{x}^M_{\{\nu+1\}} = \bar{V}^2\left(f^M - D^1\bar{x}^M_{\{\nu\}} - 3(\bar{x}^M_{\{\nu\}})^3\right)^{1/3} \tag{23.23}$$

The initial value of $\bar{x}^M_{\{0\}}$ is determined by the function

$$x_{\{0\}}(t) = -0.0875\sin 10t \tag{23.24}$$

For the parameter values $A = 500$, $B = 250$ and $C = 0$ we have, with $\omega = 10$,

$$\| P(\bar{x}^M_{\{0\}})\| = 13962.7, \qquad \| P(\bar{x}^M_{\{4\}})\| = 0.2\cdot 10^{-4}$$

$$\bar{x}^M_{\{4\}} = \begin{bmatrix} -0.389\,763\,517\cdot 10^{-2} \\ -0.520\,093\,701\cdot 10^{-1} \\ -0.801\,830\,182\cdot 10^{-1} \\ -0.877\,373\,146\cdot 10^{-1} \\ -0.774\,012\,003\cdot 10^{-1} \\ -0.463\,660\,580\cdot 10^{-1} \end{bmatrix} \tag{23.25}$$

with $\omega = 8$,

$$\| P(\bar{x}^M_{\{0\}})\| = 3837.2, \qquad \| P(\bar{x}^M_{\{4\}})\| = 0.3\cdot 10^{-4}$$

$$\begin{bmatrix} -0.656\,802\,751\cdot 10^{-2} \\ -0.816\,120\,450\cdot 10^{-1} \\ -0.125\,457\,665 \\ -0.137\,088\,885 \\ -0.120\,766\,486 \\ -0.720\,987\,737\cdot 10^{-1} \end{bmatrix} \tag{23.26}$$

with $\omega = 10/3$,

$$\| P(\bar{x}^M_{\{0\}})\| = 969.5, \qquad \| P(\bar{x}^M_{\{5\}})\| = 0.6\cdot 10^{-5}$$

$$\bar{x}^M_{\{5\}} = \begin{bmatrix} -0.510\,349\,568\cdot 10^{-1} \\ -0.480\,039\,294 \\ -0.728\,019\,121 \\ -0.790\,343\,693 \\ -0.691\,397\,770 \\ -0.405\,948\,406 \end{bmatrix} \tag{23.27}$$

For $A = 500$, $B = 250$ and $C = 20$ we obtain, with $\omega = 8$,

$$\| P(\bar{x}^M_{\{0\}}) \| = 3833.6, \qquad \| P(\bar{x}^M_{\{4\}}) \| = 0.4 \cdot 10^{-4}$$

$$\bar{x}^M_{\{4\}} = \begin{bmatrix} -0.657\,469\,244 \cdot 10^{-2} \\ -0.817\,259\,233 \cdot 10^{-1} \\ -0.125\,565\,868 \\ -0.137\,408\,767 \\ -0.120\,869\,717 \\ -0.722\,030\,794 \end{bmatrix} \tag{23.28}$$

with $\omega = 10/3$,

$$\| P(\bar{x}^M_{\{0\}}) \| = 969.9, \qquad \| P(\bar{x}^M_{\{5\}}) \| = 0.6 \cdot 10^{-5}$$

$$\bar{x}^M_{\{5\}} = \begin{bmatrix} -0.510\,847\,685 \cdot 10^{-1} \\ -0.480\,701\,463 \\ -0.728\,643\,191 \\ -0.792\,177\,623 \\ -0.691\,982\,850 \\ -0.406\,539\,252 \end{bmatrix} \tag{23.29}$$

For $A = 500$, $B = 250$ and $C = 140$ we find, with $\omega = 8$,

$$\| P(\bar{x}^M_{\{0\}}) \| = 3818.2, \qquad \| P(\bar{x}^M_{\{4\}}) \| = 0.4 \cdot 10^{-4}$$

$$\bar{x}^M_{\{4\}} = \begin{bmatrix} -0.659\,797\,820 \cdot 10^{-2} \\ -0.817\,492\,407 \cdot 10^{-1} \\ -0.124\,986\,695 \cdot 10^{-1} \\ -0.137\,837\,302 \\ -0.120\,272\,070 \\ -0.721\,928\,196 \cdot 10^{-1} \end{bmatrix} \tag{23.30}$$

with $\omega = 10/3$,

$$\| P(\bar{x}^M_{\{0\}}) \| = 998.8, \qquad \| P(\bar{x}^M_{\{5\}}) \| = 0.6 \cdot 10^{-5}$$

$$\bar{x}^M_{\{5\}} = \begin{bmatrix} -0.512\,581\,562 \cdot 10^{-1} \\ -0.480\,863\,010 \\ -0.725\,324\,099 \\ -0.794\,621\,308 \\ -0.688\,511\,638 \\ -0.406\,450\,284 \end{bmatrix} \tag{23.31}$$

The calculations presented show that a relatively small number (four or five) of steps may suffice to make a comparatively large discrepancy (of order $10^3 \div 10^5$) rather small (of order $10^{-5} \div 10^{-6}$).

EXAMPLE 23.5. Suppose it is required to find a periodic solution of period T of a nonhomogeneous equation of nth order with constant coefficients

$$\frac{d^n x}{dt^n} + c_1 \frac{d^{n-1} x}{dt^{n-1}} + \ldots + c_n x = f(t) \tag{23.32}$$

where $f(t + T) = f(t)$; and $c_i = \text{const}$.

If the imaginary roots of the characteristic equation are not multiples of the frequency in the right-hand side of $\omega = 2\pi/T$, then the trigonometric collocation method yields the following expression for the approximate value of the periodic solution of period T

$$x_m(t) = a_0 + \sum_{j=1}^{m} (a_j \cos j\omega t + b_j \sin j\omega t)$$

here

$$a_0 = \frac{\xi_0}{c_n}$$

$$a_j = \frac{\xi_j A_j - \eta_j B_j}{A_j^2 + B_j^2}$$

$$b_j = \frac{\xi_j B_j + \eta_j A_j}{A_j^2 + B_j^2} \tag{23.33}$$

where

$$A_j = \sum_{s=0}^{n}{}'' c_{n-s}(-1)^{s/2} j^s \omega^s, \quad B_j = \sum_{q=1}^{n}{}' c_{n-q}(-1)^{(q-1)/2} j^q \omega^q$$

$\sum'$ is summation over odd subscripts and $\sum''$ is summation over even ones; and ξ_j, η_j are the components of the vector

$$f^\Gamma = M f^M = (\xi_0, \xi_1, \eta_1, \ldots, \xi_m, \eta_m)$$

$$f^M = (f(t_0), f(t_1), \ldots, f(t_{2m})), \quad t_i = i \frac{T}{2m+1}$$

$$i = 1, 2, \ldots, m$$

Note that all the calculations given here, as well as in the examples below, have been carried out on an M-220 digital computer. Their realization has confirmed that it is convenient to programme the proposed formalism and that it is easy to bring it to routine form.

24. REALIZATION OF CALCULATIONS OF PERIODIC SOLUTIONS OF *T*-SYSTEMS

The use of the numerical-analytic method involves integration of periodic functions both in calculating the initial points of periodic solutions and in approximating the solutions themselves. The calculations, as has already been noted, do not cause any special difficulties when the right-hand side of a *T*-system is a polynomial with respect to the sought function.

By using the trigonometric collocation method in constructing periodic solutions of *T*-systems by the numerical-analytic method we can greatly simplify, in the general case, the calculations and adapt them for realization on a digital computer. To do this we shall seek the iterations $x_{m+1}(t, \tau, x_0)$ of the numerical-analytic scheme (4.1) in the form of the trigonometric vector polynomial of order r

$$x^r_{m+1}(t, \tau, x_0) = a_0^{m+1} \sum_{j=1}^{r} (a_j^{m+1} \cos j\omega t + b_j^{m+1} \sin j\omega t) \quad (24.1)$$

by previously expressing the integrand $f(t, x^r_m(t, \tau, x_0))$ of equality (4.1) as the interpolation trigonometric vector polynomial

$$f^r(t, x^r_m(t, \tau, x_0)) = \alpha_0^m + \sum_{j=1}^{r} (\alpha_j^m \cos j\omega t + \beta_j^m \sin j\omega t) \quad (24.2)$$

whose values at $2r + 1$ equidistant points of the interval $[0, T]$

$$t_i = i \frac{T}{2r+1}, \qquad i = 0, 1, \ldots, 2r \quad (24.3)$$

coincide with those of the function $f(t, x^r_m(t, \tau, x_0))$.

The vector of coefficients of the trigonometric polynomial (24.2)

$$f^\Gamma_m = (f^\Gamma_{1m}, f^\Gamma_{2m}, \ldots, f^\Gamma_{nm}) \quad (24.4)$$

$$f^\Gamma_{im} = (\alpha_{i0}^m, \alpha_i^m, \beta_{i1}^m, \ldots, \alpha_{ir}^m, \beta_{ir}^m) \quad (24.5)$$

$$i = 1, 2, \ldots, n$$

is expressed through the vector of values of this polynomial

$$f_m^M = (f_{1m}^M, f_{2m}^M, \ldots, f_{nm}^M) \tag{24.6}$$

$$f_{im}^M = (f_i(t_0, x_m^r(t_0, \tau, x_0)), f_i(t_1, x_m^r(t_1, \tau, x_0)),$$

$$\ldots, f_i(t_{2r}, x_m^r(t_{2r}, \tau, x_0))) \tag{24.7}$$

with the help of the matrix (21.7) by the equality

$$f_{im}^\Gamma = \Gamma f_{im}^M \qquad i = 1, 2, \ldots, n$$

The approximations $x_1^r(t, \tau, x_0), x_2^r(t, \tau, x_0), \ldots, x_{m+1}^r(t, \tau, x_0), \ldots$ to the periodic solution of a T-system, passing through the point τ, $x_0 \in D_f$, are calculated from the formulas

$$x_1^r(t, \tau, x_0) = x_0 + \int_\tau^t \left[f^r(t, x_0) - \frac{1}{T} \int_0^T f^r(t, x_0)\, dt \right] dt$$

$$x_2^r(t, \tau, x_0) = x_0 + \int_\tau^t \left[f^r(t, x_1^r(t, \tau, x_0)) \right.$$

$$\left. - \frac{1}{T} \int_0^T f^r(t, x_1^r(t, \tau, x_0))\, dt \right] dt$$

. . . . . .

$$x_{m+1}^r(t, \tau, x_0) = x_0 + \int_\tau^t \left[f^r(t, x_m^r(t, \tau, x_0)) \right.$$

$$\left. - \frac{1}{T} \int_0^T f^r(t, x_m^r(t, \tau, x_0))\, dt \right] dt \tag{24.8}$$

. . . . . .

As can be seen from (24.8), the calculation of the sequence $x_m^r(t, \tau, x_0)$ reduces to the simple operation of integrating trigonometric polynomials. To carry out the integration we can in turn use the matrix μ^1 which connects the vector of coefficients of the integral and the vector of coefficients of the integrand.

Calculations from formulas (24.8) can now be made using the scheme

$$x_1^\Gamma = x_0^\Gamma + \operatorname{diag}[\mu^1] f_0^\Gamma - z_0^\Gamma$$

$$x_2^\Gamma = x_0^\Gamma + \operatorname{diag}[\mu^1] f_1^\Gamma - z_1^\Gamma$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_{m+1}^\Gamma = x_0^\Gamma + \operatorname{diag}[\mu^1] f_m^\Gamma - z_m^\Gamma \qquad (24.9)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

where $\operatorname{diag}[\mu^1] = \operatorname{diag}[\mu^1, \ldots, \mu^1]$ is an $n(2r+1) \times n(2r+1)$-dimensional block-diagonal matrix;

$$x_j^\Gamma = (x_{1j}^\Gamma, x_{2j}^\Gamma, \ldots, x_{nj}^\Gamma), \qquad (j = 0, 1, 2, \ldots, m, \ldots)$$

x_{kj}^Γ is a vector of coefficients of the kth component of the vector function $x_j^r(t, \tau, x_0)$, $k = 1, 2, \ldots, n$;

$$z_j^\Gamma = (z_{1j}^\Gamma, z_{2j}^\Gamma, \ldots, z_{nj}^\Gamma) \quad z_{kj}^\Gamma = (z_{kj}(\tau), 0, 0, \ldots, 0)$$

$$k = 1, 2, \ldots, n$$

$z_{kj}(\tau)$ is the value, at $t = \tau$, of the trigonometric polynomial whose vector of coefficients is equal to the vector $\mu^1 f_{kj}^\Gamma$.

The following estimate is valid for the deviation of $x_m^r(t, \tau, x_0)$ from the periodic solution $x^0(t)$ of period T of the T-system

$$|x^0(t) - x_m^r(t, \tau, x_0)|$$

$$\leqslant \alpha_1(t)\left[\mathbf{Q}^m\left((E-\mathbf{Q})^{-1} + \sum_{j=1}^{3}\delta_{jm}\mathbf{B}_j\right)\mathbf{M} + \left(\sum_{i=0}^{m-1}\mathbf{Q}^i + \mathbf{B}_4\right)\mathbf{C}_r\right]$$

$$t \in [0, T] \qquad (24.10)$$

whenever the point $x^0(\tau) = x_0$ belongs to the domain D together with its $\frac{T}{2}(\mathbf{M} + \mathbf{C}_r)$ neighbourhood. Here the quantities $\alpha_1(t)$, $\mathbf{Q}$, $\mathbf{M}$, δ_{jm}, $\mathbf{B}$, $\mathbf{B}_4$ have their former meaning,

$$\mathbf{C}_r = (5 + \log r) \max_{0 \leqslant j \leqslant m-1} E_r(f(t, x_j^r(t, \tau, x_0))) \qquad (24.11)$$

$$\max_{0 \leqslant j \leqslant m-1} E_r(f(t, x_j^r(t, \tau, x_0)))$$

$$= \max_{0 \leqslant j \leqslant m-1} E_r(f_1(t, x_j^r(t, \tau, x_0))), \ldots, \max_{0 \leqslant j \leqslant m-1} E_r(f_1(t, x_j^r(t, \tau, x_0)))$$

and $E_r(f_k(t, x_j^r(t, \tau, x_0)))$ is the best uniform approximation to the function $f_k(t, x_j^r(t, \tau, x_0))$ by trigonometric polynomials of order not higher than r.

To prove the estimate (24.10) we proceed as follows. We consider the difference $x_m(t, \tau, x_0) - x_m^r(t, \tau, x_0)$. If $x_j^r(t, \tau, x_0) \in D$ for $j = 1, 2, \ldots, m$, then formulas (4.1) and (24.8) yield

$$
\begin{aligned}
&|x_j(t, \tau, x_0) - x_j^r(t, \tau, x_0)| \\
&\leqslant \alpha_1(t) |f(t, x_{j-1}^r(t, \tau, x_0)) - f^r(t, x_{j-1}^r(t, \tau, x_0))| \\
&\quad + \mathbf{K}\left[\left(1 - \frac{t-\tau}{T}\right)\int_\tau^t |x_j(s, \tau, x_0) - x_{j-1}^r(s, \tau, x_0)|\, ds\right. \\
&\quad \left. + \frac{t-\tau}{T}\int_t^{\tau+T} |x_{j-1}(s, \tau, x_0) - x_{j-1}^r(s, \tau, x_0)|\, ds\right] \qquad (24.13)
\end{aligned}
$$

for all $\tau \leqslant t \leqslant \tau + T$.

It is known [46] from interpolation theory that

$$
\begin{aligned}
&|f(t, x_{j-1}^r(t, \tau, x_0)) - f^r(t, x_{j-1}^r(t, \tau, x_0))| \\
&\qquad \leqslant (5 + \log r) E_r(f(t, x_{j-1}^r(t, \tau, x_0)))
\end{aligned}
$$

and, therefore, inequality (24.13) will become but stronger if its right-hand side augend is replaced by $(5 + \log r) E_r(f(t, x_{j-1}^r(t, \tau, x_0)))$.

Inequality (24.13) gives the following estimate for the difference under consideration

$$
\begin{aligned}
&|x_m(t, \tau, x_0) - x_m^r(t, \tau, x_0)| \\
&\qquad \leqslant [\alpha_1(t) + \mathbf{K}\alpha_2(t) + \ldots + \mathbf{K}^{m-1}\alpha_m(t)] \\
&\qquad \times (5 + \log r) \max_{0 \leqslant j \leqslant m-1} E_r(f(t, x_j^r(t, \tau, x_0))) \qquad (24.14)
\end{aligned}
$$

where $\alpha_1(t), \ldots, \alpha_m(t)$ are functions (3.3).

By Lemma 3.2 and inequality (4.12) we find from relation (24.14) that

$$
|x_m(t, \tau, x_0) - x_m^r(t, \tau, x_0)| \leqslant \alpha_1(t)\left(\sum_{i=0}^{m-1} \mathbf{Q}^i + \mathbf{B}_4\right)\mathbf{C}_r \qquad (24.15)
$$

Inequality (24.15) in conjunction with inequality (4.4) leads in an obvious way to the required estimate (24.10).

Taking into account the fact that the point $x_0 \in D$ together with its $\frac{T}{2}(\mathbf{M} + \mathbf{C}_r)$-neighbourhood it is easy to conclude that the functions $x_j^r(t, \tau, x_0)$ take values in the domain D for all $j = 1, 2, \ldots, m-1$.

This completes the proof of inequality (24.10).

It should be noted that since $E_r(f(t, x_{m-1}^r(t, \tau, x_0))) = 0(1/r)$, the sequence $x_m^r(t, \tau, x_0)$ converges uniformly to the function $x^0(t, \tau, x_0)$ when $m \to \infty$, $r \to \infty$.

Formulas (24.9) realize the numerical-analytic method with respect to the vector of coefficients of the trigonometric polynomials (24.1). The calculation process can be carried out with respect to the vectors of values of these polynomials as well. In the latter case there is no need to calculate the vector f_{im}^{Γ} and the realization of the numerical-analytic method can take place by the following scheme

$$x_1^M = x_0^M + \operatorname{diag}[V^1]\, f_0^M - z_0^M$$

$$x_2^M = x_0^M + \operatorname{diag}[V^1] f_1^M - z_1^M$$

. . . . . .

$$x_{m+1}^M = x_0^M + \operatorname{diag}[V^1] f_m^M - z_m^M \qquad (24.16)$$

where $\operatorname{diag}[V^1] = \operatorname{diag}[V^1, \ldots, V^1]$ is an $n(2r+1) \times n(2r+1)$-dimensional block-diagonal matrix;

$$x_j^M = (x_{1j}^M, x_{2j}^M, \ldots, x_{nj}^M), \qquad j = 0, 1, \ldots, m, \ldots$$

where x_{kj}^M is the vector of values of the kth component of the vector function

$$x_j^r(t, \tau, x_0), \qquad k = 1, 2, \ldots, n$$

$$z_j^M = (z_{1j}^M, z_{2j}^M, \ldots, z_{nj}^M), \qquad j = 0, 1, \ldots, m, \ldots$$

$$z_{kj}^M = (z_{kj}(\tau), z_{kj}(\tau), \ldots, z_{kj}(\tau)), \qquad k = 1, 2, \ldots, n$$

and $z_{kj}(\tau)$ is the value, at $t = \tau$, of a trigonometric polynomial whose vector of values is equal to the vector $V^1 f_{kj}^M$.

For the second-order equation

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right) \tag{24.17}$$

which is a T-system, the calculation of the periodic solution passing at $t = 0$ through the point $x_0 \in D_f$ can be carried out taking as the computed value $x^r_{m+1}(t, x_0)$ of the $(m+1)$th iteration $x_{m+1}(t, x_0)$ the trigonometric polynomial of order r of the form

$$x^r_{m+1}(t, x_0) = x_0 + \int_0^t \left\{ \int_0^t \left[f^r_m - \frac{1}{T}\int_0^T f^r_m\, dt \right] dt \right.$$
$$\left. - \frac{1}{T}\int_0^T \left[\int_0^t f^r_m - \frac{1}{T}\int_0^T f^r_m\, dt \right] dt \right\} dt \tag{24.18}$$

Here $f^r_m = f^r_m(t, x^r_m(t, \tau, x_0), dx^k_m(t, \tau, x_0)/dt)$ is a trigonometric interpolation polynomial of the form (24.2) whose values at $(2r+1)$ equidistant points of the interval $[0, T]$ coincide with those of the function $f(t, x^r_m(t, x_0), dx^r_m(t, x_0)/dt)$.

In the case under consideration the vector of coefficients x^Γ_{m+1} or the vector of values x^M_{m+1} of the trigonometric polynomial $x^\Gamma_{m+1}(t, x_0)$ is determined from the equalities

$$x^\Gamma_1 = x^\Gamma_0 + \mu^2 f^\Gamma_0 - z^\Gamma_0$$

$$x^\Gamma_2 = x^\Gamma_0 + \mu^2 f^\Gamma_1 - z^\Gamma_1$$

$$\cdots\cdots\cdots$$

$$x^\Gamma_{m+1} = x^\Gamma_0 + \mu^2 f^\Gamma_m - z^\Gamma_m \tag{24.19}$$

$$\cdots\cdots\cdots$$

$$x^M_1 = x^M_0 + V^2 f^M_0 - z^M_0$$

$$x^M_2 = x^M_0 + V^2 f^M_1 - z^M_1$$

$$\cdots\cdots\cdots$$

$$x^M_{m+1} = x^M_0 + V^2 f^M_m - z^M_m \tag{24.20}$$

where μ^2 and V^2 are the matrices (21.21) and (21.22) with $h = 1$;

$$z^\Gamma_j = (z_j(0), 0, 0, \ldots, 0), \quad j = 0, 1, 2, \ldots, m, \ldots$$

$z_j(0)$ is the value, at $t = 0$, of a trigonometric polynomial for which the vector $\mu^2 f_j^\Gamma$ is the vector of coefficients,

$$z_j^M = (z_j(0), z_j(0), \ldots, z_j(0))$$

and $z_j(0)$ is the value, at $t = 0$, of a trigonometric polynomial whose vector of values coincides with the vector $V^2 f_j^M$.

Here are some examples.

EXAMPLE 24.1. Consider a first-kind Abelian equation of the form

$$\frac{dx}{dt} = x^3 + f(t) \tag{24.21}$$

where $f(t)$ is periodic function of period T with the zero mean value

$$\overline{f(t)} = 0 \tag{24.22}$$

The periodic solution $x = x_0(t)$ of equation (24.21), if it exists, is unique. In fact, the identity

$$\frac{dx_0(t)}{dt} = x_0^3(t) + f(t) \tag{24.23}$$

yields the equality $\overline{x_0^3(t)} = 0$ which is possible only when the function $x_0(t)$ takes the values with different signs. It follows that the periodic solution of equation (24.21) takes a zero value at some point $t = \tau \in [0, T]$. Any two periodic solutions of equation (24.21) must, therefore, intersect or, by a uniqueness theorem, coincide.

So one can begin calculating the periodic solution of equation (24.21) from the value $t = \tau$, $x(\tau) = 0$. If the function $f(t)$ possesses the property

$$-f(t) = f\left(t + \frac{T}{2}\right) \tag{24.24}$$

then the periodic solution $x = x_0(t)$ of equation (24.21) also satisfies the same property

$$x_0\left(t + \frac{T}{2}\right) = -x_0(t) \tag{24.25}$$

This follows from the fact that the functions $-x_0(t)$ and $x_0(t + T/2)$ are, as one can easily see from a direct check, periodic solutions of the equation

$$\frac{dx}{dt} = x^3 - f(t) \tag{24.26}$$

and hence coincide in virtue of the uniqueness of such a solution.

The value τ at which $x_0(\tau) = 0$, for the Abel equation (24.21), (24.24), belongs to the interval $[0, T/2]$.

We find the periodic solution of equation (24.21) when

$$f(t) = 1.2\cos 10t \tag{24.27}$$

To do this, we calculate, according to the numerical-analytic method, the approximations (4.1) taking $\tau = 0$, $x(\tau) = x_0 = 0$. We count using the scheme (24.16) which assumes the following form for the equation under consideration

$$\begin{aligned} x_1^M &= V^1 f^M - z_0^M \\ x_2^M &= V^1 [(x_1^M)^3 + f^M] - z_2^M \\ &\cdots\cdots\cdots \\ x_m^M &= V^1 [(x_{m-1}^M)^3 + f^M] - z_{m-1}^M \end{aligned} \tag{24.28}$$

Here z_k^M is the vector all components of which are equal to the first element of the vector $V^1[(x_k^M)^3 + f^M]$; f^M is the vector of values of the function $f(t) = 1.2\cos 10t$; V^1 is the matrix (21.22) with $s = 1$.

The calculations have been made with $r = 15$ and $T = \pi/5$, and six approximations ($m = 6$) found. The values obtained for x_6^M and x_6^Γ are given in Table 4.

As can be seen from the values of the vector x_6^Γ,

$$\begin{aligned} x_6(t) = {} & 0.0001152 - 0.0001297\cos 10t \\ & + 0.12\sin 10t + 0.00043\cos 20t + 0.0000144\cos 30t \end{aligned} \tag{24.29}$$

to within 10^{-6}. The calculation of the Δ-constant from Δ_6 leads to the equality

$$\Delta_6(0) = 0 \tag{24.30}$$

to within 10^{-6}.

Thus the function $x_6(t)$ of the form (24.29) determines the periodic solution of equation (24.21) to within a sufficient degree of accuracy.

That such a solution exists has been established by checking inequality (7.14) of Theorem 7.2, for which purpose $x_2(t, 0, x_0)$ and $\Delta_2(x_0)$ were calculated.

Note that the values of x_2^M and those of $x_2(t, 0, 0)$ found at the same points have coincided well with each other.

TABLE 4

x_6^M	x_6^Γ
0	$0.115\,1998 \cdot 10^{-3}$
0.024 155 84	$-0.129\,6001 \cdot 10^{-3}$
0.047 323 70	0.119 9998
0.068 557 25	$0.429\,8321 \cdot 10^{-3}$
0.086 989 92	$-0.124\,4144 \cdot 10^{-6}$
0.101 8693	$0.143\,9985 \cdot 10^{-4}$
0.112 5876	$0.570\,2542 \cdot 10^{-7}$
0.118 7051	$-0.437\,5990 \cdot 10^{-10}$
0.119 9697	$-0.706\,5865 \cdot 10^{-11}$
0.116 3267	$0.183\,6447 \cdot 10^{-10}$
0.107 0228	$-0.311\,3778 \cdot 10^{-8}$
0.095 100 87	$0.111\,6966 \cdot 10^{-11}$
0.078 385 92	$0.435\,5827 \cdot 10^{-11}$
0.058 464 16	$-0.139\,6507 \cdot 10^{-12}$
0.036 153 68	$0.670\,6609 \cdot 10^{-11}$
0.012 370 61	$0.311\,5779 \cdot 10^{-11}$
−0.011 009 82	$0.784\,6613 \cdot 10^{-11}$
−0.035 693 59	$-0.563\,0112 \cdot 10^{-11}$
−0.580 0853	$-0.257\,9697 \cdot 10^{-11}$
−0.077 943 61	$0.104\,410\,07 \cdot 10^{-10}$
−0.094 685 34	$0.138\,8078 \cdot 10^{-10}$
−0.107 550 10	$-0.880\,3600 \cdot 10^{-11}$
−0.116 0115	$-0.915\,1451 \cdot 10^{-11}$
−0.119 7218	$-0.565\,3249 \cdot 10^{-11}$
−0.118 5267	$-0.101\,2296 \cdot 10^{-11}$
−0.112 4724	$-0.967\,1324 \cdot 10^{-11}$
−0.101 8047	$-0.791\,3912 \cdot 10^{-11}$
−0.086 959 95	$-0.472\,7206 \cdot 10^{-11}$
−0.068 546 81	$0.518\,7982 \cdot 10^{-12}$
−0.047 321 49	$-0.142\,3186 \cdot 10^{-11}$
−0.024 155 70	$-0.106\,1396 \cdot 10^{-11}$

EXAMPLE 24.2. Consider Duffing's equation

$$\frac{d^2x}{dt^2} + 0.36x + 2x^3 = 70 \sin 10t + \mu \tag{24.31}$$

We shall solve a periodic control problem for this equation, i. e. we shall find the value of the parameter μ at which the solution of equation (24.31), which at $t = 0$ takes the value $x(0) = 0$, is periodic. We then calculate the solution.

To do this, we shall express equation (24.31) as a system of two equations

$$\begin{aligned} \frac{dx}{dt} &= x_1 + \mu_1 \\ \frac{dx_1}{dt} &= -0.36x - 2x^3 + 70\sin 10t + \mu \end{aligned} \tag{24.32}$$

For the system of equations (24.32) formulas (24.16) take the following form

$$x_k^M = V^1 x_{1,k-1}^M - z_{10}^M$$

$$x_{1k}^M = V^1\left[-0.36x_{k-1}^M - 2(x_{k-1}^M)^3 + f^M\right] - z_{20}^M \tag{24.33}$$

$$k = 1, 2, \ldots, m, \ldots$$

where z_{10}^M is the vector each component of which is equal to the first element of the vector $V^1 x_{1k-1}^M$; z_{20}^M is the vector each component of which is equal to the first elements of the vector $V^1\left[-0.36x_{k-1}^M - 2(x_{k-1}^M)^3 + f^M\right]$; f^M is the vector of values of the function $f(t) = 70\sin 10t$.

The calculations have been carried out for $r = 15$. Table 5 gives the vectors x_4^M and x_4^Γ found for the periodic solution at the fourth step ($k = 4$). The values obtained at the fifth step coincide with those obtained at the fourth step of the iteration scheme (24.33) in all digits.

Thus $x_4(t)$ is determined by a trigonometric polynomial to within the terms of order 10^{-10}

$$x_4(t) = 0.707665\sin 10t + 0.0001906\sin 30t \tag{24.34}$$

To within the same accuracy, the value of the control parameter μ determined from $x_4(t)$ is zero. So it can be noted that the periodic solution of Duffing's equation (24.31) is determined with $\mu = 0$ to a sufficient degree of accuracy by the function (24.34).

EXAMPLE 24.3. Let us find a periodic solution of the equation

$$\frac{d^2x}{dt^2} = -\omega_0^2 \frac{x}{1 + \mu^2 x^2} + h\sin t \tag{24.35}$$

by applying the numerical-analytic method to a second-order equation.

We take $x(0) = x_0 = 0$ as the initial value of this solution, since that is the only root of the equation $\Delta_0(x_0) = 0$.

TABLE 5

x_4^M	x_4^Γ
0	$-0.4576810 \cdot 10^{-11}$
-0.1423430	$0.1270938 \cdot 10^{-11}$
-0.2788931	0.7076650
-0.4040820	$0.1589420 \cdot 10^{-10}$
-0.5127863	$-0.8951763 \cdot 10^{-11}$
-0.6005365	$-0.2000870 \cdot 10^{-10}$
-0.6637068	$0.1905555 \cdot 10^{-3}$
-0.6996755	$-0.2772984 \cdot 10^{-10}$
-0.7069450	$0.4100993 \cdot 10^{-10}$
-0.6852124	$0.1389726 \cdot 10^{-10}$
-0.6353832	$0.1945123 \cdot 10^{-10}$
-0.5595291	$-0.6094045 \cdot 10^{-11}$
-0.4607917	$-0.2701670 \cdot 10^{-10}$
-0.3432409	$-0.1741562 \cdot 10^{-11}$
-0.2116981	$-0.4035826 \cdot 10^{-10}$
$-0.7153623 \cdot 10^{-1}$	$0.1959978 \cdot 10^{-10}$
$0.7153623 \cdot 10^{-1}$	$-0.2182126 \cdot 10^{-11}$
0.2116981	$0.3202479 \cdot 10^{-10}$
0.3432409	$0.1433850 \cdot 10^{-10}$
0.4607917	$-0.6192173 \cdot 10^{-10}$
0.5595291	$-0.8175297 \cdot 10^{-10}$
0.6353832	$0.5254365 \cdot 10^{-10}$
0.6852124	$0.5343450 \cdot 10^{-10}$
0.7069450	$0.3103345 \cdot 10^{-10}$
0.6996755	$0.4983395 \cdot 10^{-11}$
0.6637068	$0.5660265 \cdot 10^{-11}$
0.6005365	$0.4713606 \cdot 10^{-10}$
0.5127863	$0.3001324 \cdot 10^{-10}$
0.4040820	$-0.1980432 \cdot 10^{-11}$
0.2788931	$0.8477736 \cdot 10^{-11}$
0.1423430	$0.7629783 \cdot 10^{-11}$

By making calculations using formulas (10.10) we obtain the expression $x_1(t, x_0) = -h \sin t$ for $x_1(t, x_0)$, and for $x_2(t, x_0)$ the expression

$$x_2(t, x_0) = -h \sin t - \frac{\omega_0^2 p}{2h} \int_0^t \ln \frac{1 + p \cos t}{1 - p \cos t} \, dt$$

where

$$p = \frac{\mu h}{(1 + \mu^2 h^2)^{1/2}}$$

As can be seen, even the finding of $x_2(t, x_0)$, let alone the subsequent approximations, causes difficulties of calculating character. The determination of $x_2(t, x_0)$ and the subsequent approximations using the scheme (24.20) allows us to avoid these difficulties.

TABLE 6

x_6^M	x_6^Γ
0	$-0.7590813 \cdot 10^{-2}$
$-0.5667896 \cdot 10^{-1}$	$0.1087192 \cdot 10^{-1}$
-0.1106463	-0.2848758
$-159\,6708$	$-0.1736431 \cdot 10^{-2}$
$-201\,3441$	$0.3368989 \cdot 10^{-2}$
-0.2351843	$-0.1563917 \cdot 10^{-2}$
-0.2654321	$-0.5018201 \cdot 10^{-4}$
-0.2853003	$-0.5438322 \cdot 10^{-3}$
-0.2928069	$-0.6954858 \cdot 10^{-3}$
-0.2880586	$0.1943995 \cdot 10^{-3}$
-0.2712353	$-0.5283720 \cdot 10^{-3}$
-0.2431839	$0.3804210 \cdot 10^{-3}$
-0.2050755	$-0.7846564 \cdot 10^{-4}$
-0.1585696	$0.1963095 \cdot 10^{-3}$
-0.1056247	$0.2063119 \cdot 10^{-3}$
$-0.4852079 \cdot 10^{-1}$	$-0.5451788 \cdot 10^{-4}$
$0.1030738 \cdot 10^{-1}$	$0.2108092 \cdot 10^{-3}$
0.683 3034	$-0.1639865 \cdot 10^{-3}$
0.123 0841	$0.5148659 \cdot 10^{-4}$
0.172 2587	$-0.1055757 \cdot 10^{-3}$
0.213 8198	$-0.9060773 \cdot 10^{-4}$
0.246 0768	$0.1739022 \cdot 10^{-4}$
0.267 7472	$-0.1135974 \cdot 10^{-3}$
0.277 9943	$0.8870169 \cdot 10^{-4}$
0.276 4432	$-0.3807339 \cdot 10^{-4}$
0.263 2017	$0.6748882 \cdot 10^{-4}$
0.238 8365	$0.4696718 \cdot 10^{-4}$
0.204 3858	$-0.3586924 \cdot 10^{-5}$
0.161 2910	$0.7078824 \cdot 10^{-4}$
0.111 4017	$-0.5397381 \cdot 10^{-7}$
$0.5683850 \cdot 10^{-1}$	$0.2995891 \cdot 10^{-4}$

In fact, for equation (24.35) the vector of values x_{k+1}^M is found from the formula

$$x_{k+1}^M = V^2\left[\frac{-\omega_0^2 x_k^M}{1+\mu^2(x_k^M)^2} + f^M\right] - z_k^M \tag{24.36}$$

where f^M is the vector of values of the function $f(t) = h\sin t$; z_k^M is the vector each component of which is equal to the first element of the vector

$$V^2\left[\frac{-\omega_0^2 x_k^M}{1+\mu^2(x_k^M)^2} + f^M\right]$$

V^2 is the matrix (21.22) with $s = 2$.

The calculations have been carried out for $r = 15$, with the following values of the parameters: $\mu = 1$, $\omega_0 = 0.6$, $h = 0.2$.

The values of x_m^M and x_m^Γ for $m = 6$ and $m = 7$ have been found. They coincide to within the fifth decimal point and are given in Table 6.

As can be seen from the table, all the harmonics, except the first, slightly dominant sinusoidal one, are equally essential for the determination of an approximate periodic solution.

25. ON THE APPLICATION OF THE TRIGONOMETRIC COLLOCATION METHOD TO THE CONSTRUCTION OF PERIODIC SOLUTIONS OF AUTONOMOUS SYSTEMS OF EQUATIONS

Both in the numerical-analytic method and the trigonometric collocation one presented above we were concerned, with few exceptions [166, 185], with nonautonomous systems of differential equations. The period of solutions sought was known *a priori* since it coincided with that of the right-hand side.

The finding of periodic solutions of autonomous systems of equations creates additional difficulties since (i) the period of such solutions is unknown and (ii) the solutions themselves are not isolated. These difficulties cause substantial obstacles in extending both methods to autonomous systems of equations.

We shall discuss the possibilities of using the calculation schemes of the trigonometric collocation method in constructing periodic solutions of autonomous systems of equations. For the sake of

simplicity, we shall restrict ourselves to a second-order equation

$$\frac{d^2x}{dt} = f\left(x, \frac{dx}{dt}\right) \tag{25.1}$$

An approximate periodic solution of the equation is sought in the form of the trigonometric polynomial

$$x_m(t) = a_0 + \sum_{j=1}^{m} (a_j \cos j\omega t + b_j \sin j\omega t) \tag{25.2}$$

where the vector of coefficients x^{Γ} and the frequency ω are unknowns. They are determined from the condition that the function $x_m(t)$ should satisfy equation (25.1) at $\tilde{N} = 2(m+1)$ equidistant points

$$\tilde{t}_i = i \frac{T}{2(m+1)} = i \frac{\pi}{\omega(m+1)} \tag{25.3}$$

$$i = 0, 1, \ldots, 2m+1$$

i. e. from the system of algebraic equations

$$\frac{d^2x_m(\tilde{t}_i)}{dt^2} = f\left(x_m(\tilde{t}_i), \frac{dx_m(\tilde{t}_i)}{dt}\right) \tag{25.4}$$

We can formalize the process of obtaining the system of equations (25.4). To do this, we introduce a rectangular matrix $\mathfrak{M}$ where the number of rows is by one larger than the number of columns. We put

$$\mathfrak{M} = \begin{bmatrix} 1 \cos\omega\tilde{t}_0 & \sin\omega\tilde{t}_0, \ldots, & \cos m\omega\tilde{t}_0 & \sin m\omega\tilde{t}_0 \\ 1 \cos\omega\tilde{t}_1 & \sin\omega\tilde{t}_1, \ldots, & \cos m\omega\tilde{t}_1 & \sin m\omega\tilde{t}_1 \\ \ldots & \ldots & \ldots & \ldots \\ 1 \cos\omega\tilde{t}_{2m} & \sin\omega\tilde{t}_{2m}, \ldots, & \cos m\omega\tilde{t}_{2m} & \sin m\omega\tilde{t}_{2m} \\ 1 \cos\omega\tilde{t}_{2m+1} & \sin\omega\tilde{t}_{2m+1}, \ldots, & \cos m\omega\tilde{t}_{2m+1} & \sin m\omega\tilde{t}_{2m+1} \end{bmatrix}$$

The elements of the matrix $\mathfrak{M} = \{\mathfrak{M}_{pq}\}$ can be expressed in the

form

$$\mathfrak{M}_{pq} = \begin{cases} 1 & \text{when } q = 1 \\ \cos(p-1)q\dfrac{\pi}{\tilde{N}} & \text{when } q = 2, 4, \ldots, 2m \\ \sin(p-1)q\dfrac{\pi}{\tilde{N}} & \text{when } q = 3, 5, \ldots, 2m+1 \end{cases} \tag{25.5}$$

$$p = 1, 2, \ldots, 2(m+1)$$

If $x^{\mathfrak{M}} = (x_m(\tilde{t}_0), x_m(\tilde{t}_1), \ldots, x_m(\tilde{t}_{2m+1}))$ is the vector of values of the trigonometric polynomial (25.2) at the points (25.3), then

$$x^{\mathfrak{M}} = \mathfrak{M}x^{\Gamma} = \mathfrak{M}\Gamma x^{M} \tag{25.6}$$

with the elements τ_{pq} of the matrix $\tau = \mathfrak{M}\Gamma$

$$\tau_{pq} = \frac{2}{2m+1}\sum_{s=1}^{m}\cos\left[2s\pi\left(\frac{p-1}{2m+2} - \frac{q-1}{2m+1}\right)\right] + \frac{1}{2m+1} \tag{25.7}$$

x^{M}, x^{Γ} and Γ have the same meaning as before, $p = 1, 2, \ldots, 2m+2$; $q = 1, 2, \ldots, 2m+1$.

The vector of values $x_{(j)}^{\mathfrak{M}}$ of the jth derivative of the function $x_m(t)$ calculated at the points (25.3),

$$x_{(j)}^{\mathfrak{M}} = \left(\frac{d^j x_m(\tilde{t}_0)}{dt^j}, \frac{d^j x_m(\tilde{t}_1)}{dt^j}, \ldots, \frac{d^j x_m(\tilde{t}_{2m+1})}{dt^j}\right)$$

is obviously connected with the vector of values x^{M} by the relation

$$x_{(j)}^{\mathfrak{M}} = \mathfrak{M}d^j x^{\Gamma} = \mathfrak{M}d^j\Gamma x^{M} \tag{25.8}$$

We obtain the elements W_{pk}^{j} of the matrices

$$W^j = \mathfrak{M}d^j\Gamma = \{W_{pk}^{j}\}$$

We first find the matrix $\tilde{L} = \mathfrak{M}I\Gamma = \{\tilde{L}_{pk}\}$. We have

$$\tilde{L}_{pk} = \frac{2}{2m+1}\sum_{s=1}^{m}\left\{\alpha_s\cos\left[\frac{(p-1)2s\pi}{2m+2} - \frac{(k-1)2s\pi}{2m+1}\right]\right.$$
$$\left. + \beta_s\sin\left[\frac{(k-1)2s\pi}{2m+1} - \frac{(p-1)2s\pi}{2m+2}\right]\right\} + \frac{\alpha_0}{2m+1} \tag{25.9}$$

Taking into account the form of the matrix d^j and relation (25.9) for the elements W^j_{pq}, we obtain the values

$$W^j_{pk} = \begin{cases} \dfrac{2}{2m+1}\omega^j \sum\limits_{s=1}^{m} (-1)^h s^j \cos\left[\dfrac{(p-1)2s\pi}{2m+2} - \dfrac{(k-1)2s\pi}{2m+1}\right] \\ \qquad\qquad \text{when } j = 2h \quad (25.10) \\ \dfrac{2}{2m+1}\omega^j \sum\limits_{s=1}^{m} (-1)^h s^j \sin\left[\dfrac{(k-1)2s\pi}{2m+1} - \dfrac{(p-1)2s\pi}{2m+2}\right] \\ \qquad\qquad \text{when } j = 2h+1 \end{cases}$$

$$j = 1, 2, \ldots, \; h = 0, 1, 2, \ldots,$$

$$p = 1, 2, \ldots, 2m+2, \; k = 1, 2, \ldots, 2m+1$$

Now the system of algebraic equations (25.4) of the trigonometric collocation method (25.2), (25.3) with respect to the vector of values x^M takes the form

$$\sum_{k=0}^{2m+1} W^2_{ik} x_m(t_k) = f\left(\sum_{k=0}^{2m+1} \tau_{ik} x_m(t_k), \quad \sum_{k=0}^{2m+1} W^1_{ik} x_m(t_k)\right) \tag{25.11}$$

$$i = 1, 2, \ldots, 2m+2$$

or, in vector form,

$$W^2 x^M = f(\tau x^M, W^1 x^M) \tag{25.12}$$

where t_k are points (19.4).

In the case when the periodic solution possesses the symmetry of the kind

$$x^0(t) = -x^0\left(t + \frac{T}{2}\right)$$

the approximate periodic solution is sought in the form (21.24) and the algebraic equations of the collocation method (25.4) are set up for $(m+2)$ equidistant points

$$\tilde{t}_i = i\frac{\pi}{\omega(m+2)}, \; i = 0, 1, \ldots, m+1 \tag{25.13}$$

These equations are written down as follows

$$\bar{W}^2\bar{x}^M = f(\bar{\tau}\bar{x}^M,\ \bar{W}^1\bar{x}^M) \tag{25.14}$$

where the elements $\bar{W}^j_{pk}$ of the matrix $\bar{W}^j = \{\bar{W}^j_{pk}\}$ are equal to the values

$$\bar{W}^j_{pk} = \begin{cases} \dfrac{2\omega^j}{m+1} \sum\limits_{s=1}^{m}{}' (-1)^h s^j \cos\left[(p-1)s\dfrac{\pi}{m+2} - (k-1)s\dfrac{\pi}{m+1}\right] & \text{when } j = 2h \\ \dfrac{2\omega^j}{m+1} \sum\limits_{s=1}^{m}{}' (-1)^h s^j \sin\left[(k-1)s\dfrac{\pi}{m+1} - (p-1)s\dfrac{\pi}{m+2}\right] & \text{when } j = 2h+1 \end{cases} \tag{25.15}$$

$$k = 1, 2, \ldots, m+1,\ p = 1, 2, \ldots, m+2,\ h = 0, 1, .2, \ldots$$

and the elements $\bar{\tau}_{pk}$ of the matrix $\bar{\tau} = \{\bar{\tau}_{pk}\}$ are equal to the values

$$\bar{\tau}_{pk} = \frac{2}{m+1} \sum_{s=1}^{\frac{m+1}{2}} \cos\left[(2s-1)\pi\left(\frac{p-1}{m+2} - \frac{k-1}{m+1}\right)\right] \tag{25.16}$$

$$k = 1, 2, \ldots, m+1,\ p = 1, 2, \ldots, m+2$$

Solving a system of algebraic equations of the collocation method, we find both the frequency of a periodic solution and its approximate expression. We are not dealing here with questions of solvability of these equations or convergence of the approximate values ω_m and x^M to the exact values of the frequency ω and the periodic solution $x = x^0(t)$. These questions justify applying the trigonometric collocation method to the finding of periodic solutions of autonomous systems and can be answered positively on the basis of the general theorems recently derived by Stokes [181, 182] and Strygin [187].

To illustrate the calculation scheme, we shall find a periodic solution of the Van der Pol equation

$$\frac{d^2x}{dt^2} - \varepsilon(1-x^2)\frac{dx}{dt} + x = 0 \tag{25.17}$$

The system of algebraic equations (25.14) with respect to the vector of values $\bar{x}^M = (\bar{x}_m(t_0), \bar{x}_m(t_1), \dots \bar{x}_m(t_m))$ has the form

$$\overline{Wx}^M - \varepsilon(1 - (\bar{x}^M)^2)\,\bar{W}^1\bar{x}^M + \overline{\tau x}^M = 0 \qquad (25.18)$$

For $\varepsilon = 0.2$ and $m = 23$ the system of algebraic equations (25.18) has been solved by the Newton-Kantorovich method. The column ($\omega_0 = 6.28$) of Table 7 gives the initial approximation to the frequency ω_0 and the vector of values $\bar{x}_0^M$, and the column ($\omega_{\{18\}} = 0.101020 \cdot 10^1$) gives the approximate value of the frequency $\omega_{\{18\}}$ and the vector $\bar{x}_{\{18\}}^M$ obtained at the 18th step. The discrepancy of the initial approximation is of order 10^5 and that of the approximation found is of order 10^{-1}.

TABLE 7

$\omega_0 = 6 \cdot 28$	$\omega_{\{18\}} = 0 \cdot 101020 \cdot 10^1$	$\omega_0 = 6 \cdot 28$	$\omega_{\{18\}} = 0 \cdot 101020 \cdot 10^1$
0	$-0.603\,936$	-2.05	$-0.184\,429 \cdot 10^1$
-0.5	$-0.863\,926$	-2.0	$-0.174\,064 \cdot 10^1$
-0.6	$-0.111\,066 \cdot 10^1$	-1.8	$-0.161\,329 \cdot 10^1$
-0.7	$-0.133\,673 \cdot 10^1$	-1.7	$-0.146\,405 \cdot 10^1$
-0.75	$-0.153\,532 \cdot 10^1$	-1.6	$-0.129\,448 \cdot 10^1$
-0.8	$-0.170\,091 \cdot 10^1$	-1.5	$-0.110\,586 \cdot 10^1$
-0.9	$-0.182\,986 \cdot 10^1$	-1.4	$-0.899\,417$
-1.0	$-0.192\,053 \cdot 10^1$	-1.2	$-0.676\,457$
-1.2	$-0.197\,313 \cdot 10^1$	-0.9	$-0.438\,620$
-1.4	$-0.198\,922 \cdot 10^1$	-0.6	$-0.188\,093$
-1.8	$-0.197\,125 \cdot 10^1$	-0.3	$0.720\,854 \cdot 10^{-1}$
-2.1	$-0.192\,201 \cdot 10^1$	-0.1	$0.337\,858$

As can be seen from the table, the frequency of the obtained periodic solution is close to one and its amplitude to two, which is in good agreement with the conclusions concerning the frequency and amplitude of the periodic solution of the Van der Pol equation obtained by an asymptotic method for small values of ε [16].

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